Any Shape Can Ultimately Cross Information on Two-Dimensional Abelian Sandpile Models

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Abstract. We study the abelian sandpile model on the two-dimensional grid with uniform neighborhood (a number-conserving cellular automata), and prove that any family of discrete neighborhoods defined as scalings of a continuous non-flat shape can ultimately perform crossing.
Keywords. Sandpile models, crossing information, prediction problem.

1 Introduction

In [1], three physicists proposed the now famous two-dimensional *abelian sandpile model* with von Neumann neighborhood of radius one. This number-conserving discrete dynamical system is defined by a simple local rule describing the movements of sand grains in the discrete plane $\mathbb{Z}^2$, and exhibits surprisingly complex global behaviors.

The model has been generalized to any directed graph in [2,3]. Basically, given a digraph, each vertex has a number of sand grains on it, and a vertex that has more grains than out-neighbors can *fire* and give one grain to each of its out-neighbors. This model is Turing-universal [8]. When restricted to particular directed graphs (digraphs), an interesting notion of complexity is given by the following *prediction problem*.

> **Prediction problem.**
> *Input:* a finite and stable configuration, and two vertices u and v.
> *Question:* does adding one grain on vertex u trigger a chain of reactions that will reach vertex v?

The computational complexity in time of this problem has been proven to be P-hard or in NC (solvable in polylogarithmic time on a parallel machine with a polynomial number of processors), depending on the restrictions applied to the digraph [11]. In order to prove the P-hardness of the prediction problem, authors naturally try to implement circuit computations, via reductions from the *Monotone Circuit Value Problem* (MCVP), *i.e.*, they show how to implement the following set of gates: *wire*, *turn*, *multiply*, *and*, *or*, and *crossing*.

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In abelian sandpile models, monotone gates are usually easy to implement with *wires* constructed from sequences of vertices that fire one after the other[3]: an *or* gate is a vertex that needs one of its in-neighbors to fire; an *and* gate is a vertex that needs two of its in-neighbors to fire. The crucial part in the reduction is therefore the implementation of a *crossing* between two wires. Regarding regular graphs, the most relevant case is the two-dimensional grid (in dimension one crossing is less meaningful, and from dimension three it is easy to perform a crossing using an extra dimension; see Section 3 for references).

When it is possible to implement a crossing, then the prediction problem is P-hard. The question is now to formally relate the impossibility to perform a crossing with the computational complexity of the prediction problem. The goal is thus to find conditions on a neighborhood so that it cannot perform a crossing (this requires a precise definition of *crossing*), and prove that these conditions also imply that the prediction problem is in NC. As a hint for the existence of such a link, it is proven in [7] that crossing information is not possible with von Neumann neighborhood of radius one, for which the computational complexity of the prediction problem has not yet been proven to be P-hard (neither in NC). The present work continues the study on general uniform neighborhoods, and shows that the conditions on the neighborhood so that it can or cannot perform crossing are intrinsically discrete: any *shape* of neighborhood (in $\mathbb{R}^2$, see Section 2) can perform crossing (Theorem 2).

Section 2 defines the abelian sandpile model, neighborhood, shape, and crossing configuration (this last one requires a substantial number of elements to be defined with precision, as it is one of our aims), and Section 3 reviews the main known results related to prediction problem and information crossing. The notion of firing graph (from [7]) is presented and studied at the beginning of Section 4, which then establishes some conditions on crossing configurations for convex neighborhoods, and finally exposes the main result of this paper: that any shape can ultimately perform crossing.

2 Definitions

In the literature, *abelian sandpile model* and *chip-firing game* usually refer to the same discrete dynamical system, sometimes on different classes of (un)directed graphs.

2.1 Abelian sandpile models on $\mathbb{Z}^2$ with uniform neighborhood

Given a digraph $G = (V, A)$, we denote $d^+(v)$ (resp. $d^-(v)$) the out-degree (resp. in-degree) of vertex $v \in V$, and $\mathcal{N}^+(v)$ (resp. $\mathcal{N}^-(v)$) its set of out-neighbors (resp. in-neighbors). A *configuration* c is an assignment of a finite number of sand grains to each vertex, $c : V \to \mathbb{N}$. The global rule $F : \mathbb{N}^{|V|} \to \mathbb{N}^{|V|}$ is defined

[3] this is a particular case of *signal* (*i.e.*, information transport) that we can qualify as *elementary*.

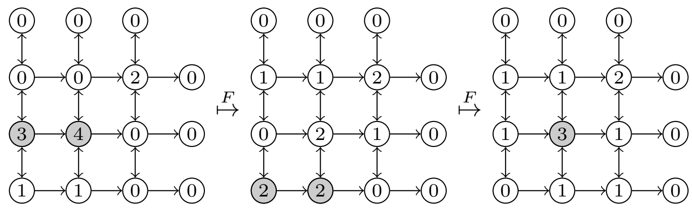


Fig. 1: An example of two evolution steps in the abelian sandpile model.

by the parallel application of a local rule at each vertex: if vertex v contains at least $d^+(v)$ grains, then it gives one grain to each of its out-neighbors (we say that v *fires*, or v is a *firing* vertex). Formally,

$$\forall v \in V,\ \big(F(c)\big)(v) = c(v) - d^+(v)\mathbf{1}_{\mathbb{N}}\big(c(v) - d^+(v)\big) + \sum_{u \in \mathcal{N}^-(v)} \mathbf{1}_{\mathbb{N}}\big(c(u) - d^+(u)\big) \tag{1}$$

with $\mathbf{1}_{\mathbb{N}}(x)$ the indicator function of $\mathbb{N}$, that equals 1 when $x \geq 0$ and 0 when $x < 0$. Note that this discrete dynamical system is deterministic (example on Figure 1).

Remark 1. As self-loops (arcs of the form (v, v) for some $v \in V$) are not useful for the dynamics (it just "traps" some grains on vertices), all our digraphs will be loopless.

A vertex v is *stable* when $c(v) < d^+(v)$, and *unstable* otherwise. By extension, a configuration c is *stable* when all the vertices are stable, and *unstable* if at least one vertex is unstable. Given a configuration c, we denote $Stab(c)$ (resp. $Act(c)$) the set of stable (resp. unstable) vertices.

In this work, we are interested in the dynamics when vertices are embedded in the plane at integer coordinates $\mathbb{Z}^2$, with a uniform neighborhood. In mathematical terms, given some finite *neighborhood* $\mathcal{N}^+ \subset \mathbb{Z}^2$, we define the graph $G^{\mathcal{N}^+} = (V, A^{\mathcal{N}^+})$ with $V = \mathbb{Z}^2$ and

$$A^{\mathcal{N}^+} = \big\{\big((x, y), (x', y')\big) \mid (x' - x, y' - y) \in \mathcal{N}^+\big\}. \tag{2}$$

On $G^{\mathcal{N}^+}$ a vertex fires if it has at least $p^{\mathcal{N}^+} = |\mathcal{N}^+|$ grains. When there is no ambiguity, we will omit the superscript $\mathcal{N}^+$ for simplicity. An example is given on Figure 2.

We say that a configuration is *finite* when it contains a finite number of grains, or equivalently when the number of non-empty vertices is finite (by definition, the number of grains on each vertex is finite). We say that a finite configuration c is a square of size $n \times n$ if there is no grain outside a window of size n by n cells: there exists (x_0, y_0) such that for all $(x, y) \in \mathbb{Z}^2 \setminus \{(x', y') \mid (x_0 \leq x' < x_0 + n) \wedge (y_0 \leq y' < y_0 + n)\}$ we have $c((x, y)) = 0$.

Definition 1 (movement vector). *Given a* neighborhood $\mathcal{N}^+ \subset \mathbb{Z}^2 \setminus \{(0, 0)\}$ *of p cells, $\overrightarrow{v} \in \mathcal{N}^+$ is called a* movement vector*. We denote $\mathcal{N}^+(u) = \mathcal{N}^+ + \overrightarrow{u}$*

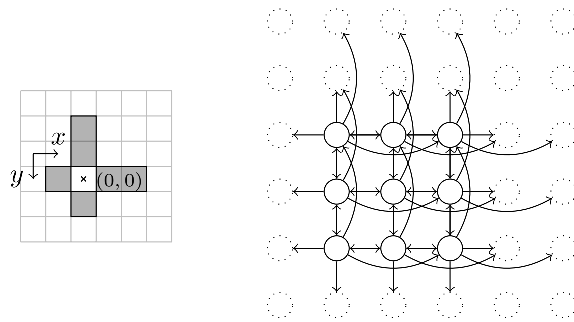


Fig. 2: A neighborhood $\mathcal{N}^+$ (left) and a part of the corresponding graph $G^{\mathcal{N}^+}$ (right), $p = 6$.

the set of neighbors of u. As we work on $\mathbb{Z}^2$ we always assume that there are at least two non-collinear movement vectors.

We will only study finite neighborhoods and finite configurations, which ensures that the dynamics converges when the graph is connected (potential energy dissipates). Finally, there is a natural notion of addition among configurations c, c' on the same set of vertices V, defined as $(c + c')(v) = c(v) + c'(v)$ for all $v \in V$.

2.2 Shape of neighborhood

A shape will be defined as a continuous area in $\mathbb{R}^2$, that can be placed on the grid to get a discrete neighborhood $\mathcal{N}^+$ that defines a graph $G^{\mathcal{N}^+}$ for the abelian sandpile model.

Definition 2 (shape). *A* shape *(at $(0,0)$) is a bounded set $s^+ \subset \mathbb{R}^2$. The* neighborhood $\mathcal{N}^+_{s^+,r}$ of shape s^+ *(with the firing cell at $(0,0)$)* with scaling ratio $r \in \mathbb{R}$*, $r > 0$, as*

$$\mathcal{N}^+_{s^+,r} = \{(x, y) \in \mathbb{Z}^2 \mid (x/r, y/r) \in s^+\} \setminus \{(0, 0)\}.$$

We also have movement vectors $\vec{v} \in s^+$*, and denote $s^+(v) = s^+ + \vec{v}$.*

We recall Remark 1: self-loops are removed from the dynamics. A shape is bounded so that its corresponding neighborhoods are finite (*i.e.*, there is a finite number of neighbors). An example of shape is given on Figure 3.

Remark that a given neighborhood $\mathcal{N}^+ \subset \mathbb{Z}^2$ always corresponds to an infinity of pairs $\langle$shape, scaling ratio$\rangle$. The notion of inverse shape and inverse neighborhood will be of interest in the analysis of Section 4: it defines the set of cells which have a given cell in their neighborhood (the neighboring relation is not symmetric).

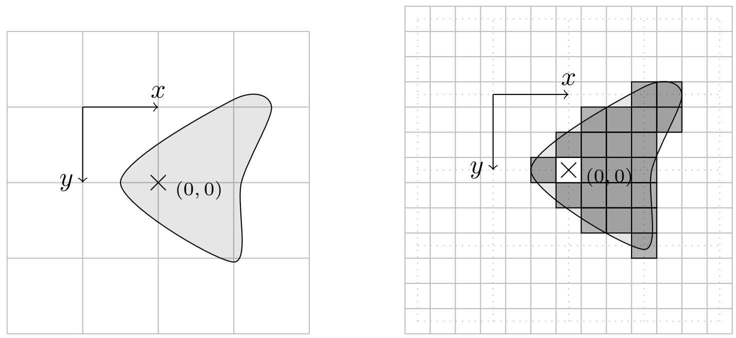


Fig. 3: A shape s^+ on $\mathbb{R}^2$ (left), and the neighborhood $\mathcal{N}^+_{s^+,3}$ (right, dotted lines reproduce the original grid from the left picture, and the discrete neighborhood in $\mathbb{Z}^2$ is darken).

Definition 3 (inverse). *The inverse* $\mathcal{N}^-$ *(resp.* s^-*) of a neighborhood* $\mathcal{N}^+$ *(resp. of a shape* s^+*) is defined via the central symmetry around* $(0,0)$*,*

$$\mathcal{N}^- = \left\{(x,y) \in \mathbb{Z}^2 \mid (-x,-y) \in \mathcal{N}^+\right\} \text{ and } s^- = \left\{(x,y) \in \mathbb{R}^2 \mid (-x,-y) \in s^+\right\}.$$

Remark 2. For any shape s^+ and any ratio $r > 0$, we have $\mathcal{N}^-_{s^+,r} = \mathcal{N}^+_{s^-,r}$.

We also have the inverse shape $s^-(v)$ at any point $v \in \mathbb{R}^2$ and the inverse neighborhood $\mathcal{N}^-(v)$ at any point $v \in \mathbb{Z}^2$. For any $u, v \in \mathbb{Z}^2$ (resp. $\mathbb{R}^2$),

$$v \in \mathcal{N}^+(u) \iff u \in \mathcal{N}^-(v) \qquad (\text{resp. } v \in s^+(u) \iff u \in s^-(v)).$$

We want shapes to have some thickness everywhere, as stated in the next definition. We denote $T_{(x,y),(x',y'),(x'',y'')}$ the triangle of corners $(x,y), (x',y'), (x'',y'') \in \mathbb{R}^2$.

Definition 4 (non-flat shape). *A shape* s^+ *is* non-flat *when for every point* $(x,y) \in s^+$ *there exist* $(x',y'), (x'',y'') \in \mathbb{R}^2$ *such that the triangle* $T_{(x,y),(x',y'),(x'',y'')}$ *has a strictly positive area (*i.e.*, the three points are not aligned), and entirely belongs to* s^+*.*

2.3 Crossing configuration

The following definitions are inspired by [7]. A *crossing configuration* will be a finite configuration, and for convenience with the definition we take it of size $n \times n$ for some $n \in \mathbb{N}$, with non-empty vertices inside the square from $(0,0)$ to $(n-1, n-1)$ (see Figure 4). The idea is to be able to add a grain on the West border to create a

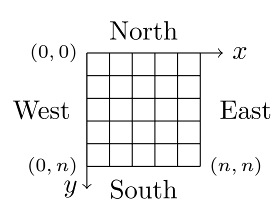


Fig. 4: Orientation and positioning of an $n \times n$ square.

chain of reactions that reaches the East border, and a grain on the North border to create a chain of reactions that reaches the South border.

Let $E_n \subset \{0,1\}^n$ be the set of vectors $\overrightarrow{e_i}$, where $\overrightarrow{e_i}$ has its i^{th} component as 1, and all the other components as 0. That is $E_n = \{\overrightarrow{e_i} \in \{0,1\}^n \mid \overrightarrow{e_i}(i) = 1 \text{ and } \overrightarrow{e_i}(j) = 0 \text{ for } j \neq i\}$.

In order to convert vectors to configurations, we define four positions of a given vector $\overrightarrow{e} \in \{0,1\}^n$: $N(\overrightarrow{e})$, $W(\overrightarrow{e})$, $S(\overrightarrow{e})$ and $E(\overrightarrow{e})$ are four configurations of size $n \times n$, defined as

$$N(\overrightarrow{e}) : (x,y) \mapsto \begin{cases} \overrightarrow{e}(x) & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases} \qquad E(\overrightarrow{e})) : (x,y) \mapsto \begin{cases} \overrightarrow{e}(y) & \text{if } x = n-1 \\ 0 & \text{otherwise} \end{cases}$$

$$S(\overrightarrow{e}) : (x,y) \mapsto \begin{cases} \overrightarrow{e}(x) & \text{if } y = n-1 \\ 0 & \text{otherwise} \end{cases} \qquad W(\overrightarrow{e}) : (x,y) \mapsto \begin{cases} \overrightarrow{e}(y) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

The idea is for example that $c + W(\overrightarrow{w})$, represents the configuration obtained by the addition of one grain to c on the cell of the West border given by $\overrightarrow{w} \in E_n$.

Definition 5 (transporter). *A finite configuration c of size $n \times n$ is a* transporter from West to East with vectors $\overrightarrow{w}, \overrightarrow{e} \in E_n$ *when*

1. *c is stable;*
2. *$\exists t \in \mathbb{N},\ Act(F^t(c + W(\overrightarrow{w}))) = \{v \in \mathbb{Z}^2 \mid E(\overrightarrow{e})(v) = 1\}$.*

Symmetrically, c is a transporter from North to South with vectors $\overrightarrow{n}, \overrightarrow{s} \in E_n$ *when*

1. *c is stable;*
2. *$\exists t \in \mathbb{N},\ Act(F^t(c + N(\overrightarrow{n}))) = \{v \in \mathbb{Z}^2 \mid S(\overrightarrow{s})(v) = 1\}$.*

Let us recall the Abelian property of sandpile models [3], which implies that the order of firings has no importance, hence our focus on the set *Act*. Besides transport of a signal (implemented via firings) from one border to the other (from West to East, and from North to South), a proper crossing of signals must not fire any cell on the other border: the transport from West to East must not fire any cell on the South border, and the transport from North to South must not fire any cell on the East border. This is the notion of isolation.

Definition 6 (isolation). *A finite configuration c of size $n \times n$ has* West vector $\overrightarrow{w} \in E_n$ isolated to the South *when*

1. *$\forall t \in \mathbb{N},\ Act(F^t(c + W(\overrightarrow{w}))) \cap \{(x,y) \mid y = n-1\} = \emptyset$.*

Symmetrically, c has North vector $\overrightarrow{n} \in E_n$ isolated to the East *when*

1. *$\forall t \in \mathbb{N},\ Act(F^t(c + N(\overrightarrow{n}))) \cap \{(x,y) \mid x = n-1\} = \emptyset$.*

Definition 7 (crossing configuration). *A finite configuration c of size $n \times n$ is a* crossing with vectors $\overrightarrow{n}, \overrightarrow{e}, \overrightarrow{s}, \overrightarrow{w} \in E_n$ *when*

1. *c is stable;*

2. *c is a transporter from West to East with vectors* $\vec{w}, \vec{e}$*;*
3. *c has West vector* $\vec{w}$ *isolated to the South;*
4. *c is a transporter from North to South with vectors* $\vec{n}, \vec{s}$*;*
5. *c has North vector* $\vec{n}$ *isolated to the East.*

Definition 8 (neighborhood crossing). *A neighborhood* $\mathcal{N}^+$ can perform crossing *if there exists a crossing configuration in the abelian sandpile model on* $G^{\mathcal{N}^+}$*.*

Definition 9 (shape ultimately crossing). *A shape* s^+ can ultimately perform crossing *if there exists a ratio* $r_0 \in \mathbb{R}$ *such that for all* $r \in \mathbb{R}$*,* $r \geq r_0$*, the neighborhood* $\mathcal{N}^+_{s^+,r}$ *can perform crossing.*

As mentioned at the beginning of this subsection, the definition of crossing configuration can be generalized as follows.

Remark 3. Crossings can be performed in different orientations (not necessarily from the North border to the South border, and from the West border to the East border), the important property of the chosen borders is that the crossing comes from two *adjacent borders*, and escapes toward the two *mirror borders* (the mirror of North being South, the mirror of West being East, and reciprocally). It can also be delimited by a rectangle of size $n \times m$ for some integers n and m, instead of a square.

Adding one grain on a border of some stable configuration ensures that the dynamics converges in linear time in the size of the stable configuration.

Lemma 1. *Let c be a finite stable configuration of size* $n \times m$*, then for any* $\vec{w} \in E_n$*, every vertex is fired at most once during the evolution from* $c + W(\vec{w})$ *to a stable configuration.*

3 Related results on prediction problems

As mentioned in the introduction, proofs of P-hardness via reductions from MCVP relate the ability to perform crossing to the computational complexity of the prediction problem.

Let us first mention that Tardos proved in [13] that for undirected sandpiles (corresponding to symmetric neighorhoods in our setting), the prediction problem is solvable in polynomial time.

Regarding the classical neighborhoods of von Neumann (in dimension d each cells has $2d$ neighbors corresponding to the two direct neighbors in each dimension, for example in dimension two the four neighbors are the North, East, South, and West cells) and Moore (von Neumann plus the diagonal cells), it is known that the prediction problem is in NC in dimension one and P-hard in dimension at least three [12] (via a reduction from MCVP in which it is proven that they can perform crossing). Whether their prediction problem is in NC or P-hard in dimension two is an open question, though we know that they cannot perform crossing [7].

More general neighborhoods have also been studied, such as Kadanoff sandpile models for which it has been proven that the prediction problem is in NC in dimension one [4] (improved in [5] and generalized to any decreasing sandpile model in [6]), and P-hard in dimension two when the radius is at least two (via a reduction from MCVP in which it is proven that it can perform crossing).

Threshold automata (including the majority cellular automata on von Neumann neighborhood in dimension two, which prediction problem is also not known to be in NC or P-hard) are closely related, it has been proven that it is possible to perform crossing on undirected planar graphs of degree at most five [10], hence hinting that degree four regular graph, *i.e.*, such that $V = \mathbb{Z}^2$, is the most relevant case of study. The link between the ability to perform crossing and the P-hardness of the prediction problem has been formally stated in [9].

4 Study of neighborhood, shape and crossing

4.1 Distinct firing graphs

A firing graph is a useful representation of the meaningful information about a crossing configuration: which vertices fire, and which vertices trigger the firing of other vertices.

Definition 10 (firing graph, from [7]). *Given a stable configuration c and a vertex v on which we add one grain, the* firing graph *of this chain of reaction is the digraph $G = (V, A)$ with:*

- *V is the set of all fired vertices;*
- *there is an arc $(v_1, v_2) \in A$ when $v_1, v_2 \in V$ and v_1 is fired strictly before v_2.*

Remark 4. To a crossing configuration c with vectors $\overrightarrow{n}, \overrightarrow{e}, \overrightarrow{s}, \overrightarrow{w}$, we associate two *firing graphs* G_{we}, G_{ns}, where G_{we} (resp. G_{ns}) is the firing graph relative to the grain addition given by $\overrightarrow{w}$ (resp. $\overrightarrow{n}$).

In this section we make some notations a little more precise, by subscripting the degree and set of neighbors with the digraph it is relative to. For example $d^+_G(v)$ denotes the out-degree of vertex v in digraph G. The following result is correct on all Eulerian digraph G (*i.e.*, a digraph such that $d^+_G(v) = d^-_G(v)$ for all vertex v), which includes the case of a uniform neighborhood on the grid $\mathbb{Z}^2$.

Theorem 1. *Given an Eulerian digraph G for the abelian sandpile model, if there exists a crossing then there exists a crossing with firing graphs $G'_1 = (V'_1, A'_1)$ and $G'_2 = (V'_2, A'_2)$ such that $V'_1 \cap V'_2 = \emptyset$.*

Proof (sketch). The proof is constructive and follows a simple idea: if a vertex is part of both firing graphs, then it is not useful to perform the crossing, and we can remove it from both firing graphs. Let c be a crossing configuration, and $G_1 = (V_1, A_1), G_2 = (V_2, A_2)$ its two firing graphs. We will explain how to construct a configuration c' such that the respective firing graphs $G'_1 = (V'_1, A'_1)$ and $G'_2 = (V'_2, A'_2)$ verify:

- $V_1' = V_1 \setminus (V_1 \cap V_2)$
- $V_2' = V_2 \setminus (V_1 \cap V_2)$.

This ensures that $V_1' \cap V_2' = \emptyset$, the expected result.

Construction. The construction applies two kinds of modifications to the original crossing c: it removes all the grains from vertices in the intersection of G_1 and G_2 so that they are not fired anymore, and adds more sand to their out-neighbors so that the remaining vertices are still fired. Formally, the configuration c' is identical to the configuration c, except that:

- for all $v \in V_1 \cap V_2$ we set $c'(v) = 0$;
- for all $v \in \left(\bigcup_{v \in V_1 \cap V_2} \mathcal{N}^+_{G_1}(v) \right) \setminus \left(\bigcup_{v \in V_1 \cap V_2} \mathcal{N}^+_{G_2}(v) \right)$,
  we set $c'(v) = c(v) + |\mathcal{N}^-_{G_1}(v) \cap (V_1 \cap V_2)|$;
- for all $v \in \left(\bigcup_{v \in V_1 \cap V_2} \mathcal{N}^+_{G_2}(v) \right) \setminus \left(\bigcup_{v \in V_1 \cap V_2} \mathcal{N}^+_{G_1}(v) \right)$,
  we set $c'(v) = c(v) + |\mathcal{N}^-_{G_2}(v) \cap (V_1 \cap V_2)|$.

In order to prove that c' is such that its two firing graphs G_1' and G_2' verify the two claims, we combine the following three facts.

Fact 1. No new vertex is fired: $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$.

Fact 2. The vertices of $V_1 \cap V_2$ are not fired in G_1' nor G_2':

$$V_1' \cap (V_1 \cap V_2) = \emptyset \text{ and } V_2' \cap (V_1 \cap V_2) = \emptyset.$$

Fact 3. The vertices of V_1 (resp. V_2) which do not belong to $V_1 \cap V_2$ are still firing in G_1' (resp. G_2'):

$$V_1 \setminus (V_1 \cap V_2) \subseteq V_1' \text{ and } V_2 \setminus (V_1 \cap V_2) \subseteq V_2'.$$

Conclusion. Let us argue that c' is indeed a crossing configuration. It is stable by construction because the amount of added grains cannot create instabilities ($|\mathcal{N}^-_{G_i}(v) \cap (V_1 \cap V_2)| < p - c(v)$ by definition of firing graphs); it is isolated because G_1' and G_2' are subgraphs of respectively G_1 and G_2 (Fact 1) which were isolated; it is a transporter because G_1' and G_2' are firing graphs and vertices on the North, East, South and West borders cannot belong to $V_1 \cap V_2$, therefore (Fact 3) G_1' and G_2' still connect two adjacent borders to the two mirror borders. $\square$

We can restate Theorem 1 as follows: if crossing is possible, then there exists a crossing with two firing graphs which have no common firing cells. It is useful to prove that some neighborhoods (of small size p) cannot perform crossing, such as the impossibility of crossing with von Neumann and Moore neighborhoods of radius one, which was proved in [7].

Corollary 1 ([7]). *Von Neumann and Moore neighborhoods of radius one cannot cross.*

Proof (Alternative proof). A combinatorial study of these two neighborhoods straightforwardly leads to the impossibility of having two firing graphs that cross each other (at least two respective arcs intersect) and have disjoint sets of vertices. □

4.2 Convex shapes and neighborhoods

Theorem 1 is also convenient to give constraints on crossing configurations for some particular family of neighborhoods.

Definition 11 (Convex shape and neighborhood). *A shape s^+ is* convex *if and only if for any $u, v \in s^+$, the segment from u to v also belongs to s^+: $[u, v] \subset s^+$. A neighborhood $\mathcal{N}^+$ is* convex *if and only if there exists a convex shape s^+ and ratio $r > 0$ such that $\mathcal{N}^+_{s^+,r} = \mathcal{N}^+$.*

In the design of crossing configurations, it is natural to try the simpler case first: put $p-1$ grains on vertices we want to successively fire, and no grain on other vertices. The following corollary states that this simple design does not work if the neighborhood is convex.

Corollary 2. *For a convex neighborhood with a shape s^+ containing $(0,0)$, a crossing configuration c must have at least one firing vertex v such that $c(v) \leq p-2$ grains.*

Proof. Let us consider a crossing configuration c with two firing graphs $G_1 = (V_1, A_1)$, $G_2 = (V_1, A_1)$. According to Theorem 1 and its construction, there exist two disjoint firing graphs $G'_1 = (V'_1, A'_1) \subseteq G_1$, $G'_2 = (V'_2, A'_2) \subseteq G_2$. Then, any pair of crossing arcs between the two subgraphs is a pair of crossing arcs between G_1, G_2. Consider one of such pairs, say $((h_1, h_2), (v_1, v_2))$, where $h_1, h_2 \in V'_1 \subseteq V_1$ and $v_1, v_2 \in V'_2 \subseteq V_2$. Since the neighborhood is convex and $(0,0) \in s^+$, either h_2 is a neighbor of v_1, or v_2 is a neighbor of h_1. Assume that h_2 is a neighbor of v_1, as $h_2 \in V'_1 \subseteq V_1$ then $h_2 \notin (V_1 \cap V_2)$, so $h_2 \notin V_2$. It means that in configuration c, firing v_1 does not fire h_2, hence $c(h_2) < p-1$. □

4.3 Crossing and shapes

In this section we prove our main result: any shape can ultimately perform crossing. We first analyse how regions inside a shape scale with r. The following lemma is straightforward from the definition of the neighborhood of a shape (Definition 2), it expresses the fact that neighboring relations are somehow preserved when we convert shapes to neighborhoods.

Lemma 2. *Let $s^{+1}, \ldots, s^{+k} \subset \mathbb{R}^2$ be a partition of the shape s^+, then $\mathcal{N}^+_{s^{+1},r}, \ldots, \mathcal{N}^+_{s^{+k},r}$ is a partition of the neighborhood $\mathcal{N}^+_{s^+,r}$.*

The next lemma states that any non-flat region inside a shape can be converted (with some appropriate ratio) to an arbitrary number of discrete cells in the corresponding neighborhood. The proof is also straightforward from the definition of non-flat shapes.

Lemma 3. *Let s^+ be a shape, and $s' \subseteq s^+$ be non-empty and non-flat. Then for any $k \in \mathbb{N}$, there exists a ratio $r_0 > 0$ such that for any $r \geq r_0$, $|\mathcal{N}^+_{s',r}| \geq k$.*

Remark 5. Lemmas 2 and 3 also apply to the inverse shape s^- and the inverse neighborhood $\mathcal{N}^-_{s^+,r}$, because the inverse neighborhood (resp. shape) is also a neighborhood (resp. shape).

Let us state a useful consequence of Lemmas 2 and 3, saying that neighborhing relations of non-empty non-flat regions (in $\mathbb{R}^2$) can be preserved by discretization (in $\mathbb{Z}^2$).

Lemma 4. *Let s^+ be a shape. Given points $p_1, \ldots, p_\ell \in \mathbb{R}^2$, $\epsilon > 0$ and $k_1, \ldots, k_\ell \in \mathbb{N}$, there exists $r_0 > 0$ such that for any $r \geq r_0$ we have discrete sets $S_1, \ldots, S_\ell \subseteq \mathbb{Z}^2$ respectively corresponding to $p_1, \ldots, p_\ell$, with $|S_i| \geq k_i$ for all i, and ($B_\epsilon(p)$ is the ball of radius ϵ around p)*

$$B_\epsilon(p_j) \subseteq \bigcap_{p \in B_\epsilon(p_i)} (p + s^+) \implies S_j \subseteq \bigcap_{v \in S_i} \mathcal{N}^+_{s^+,r}(v)$$
$$B_\epsilon(p_j) \cap \bigcap_{p \in B_\epsilon(p_i)} (p + s^+) = \emptyset \implies S_j \cap \bigcap_{v \in S_i} \mathcal{N}^+_{s^+,r}(v) = \emptyset.$$

In other words, if all points in the ball $B_\epsilon(p_i)$ have ball $B_\epsilon(p_j)$ entirely in their neighborhood according to s^+, then all vertices of the set S_i have all vertices of the set S_j in their neighborhood according to $\mathcal{N}^+_{s^+,r}$; and if no point in the ball $B_\epsilon(p_i)$ has any point of the ball $B_\epsilon(p_j)$ in its neighborhood according to s^+, then no vertex of the set S_i has any vertex of the set S_j in its neighborhood according to $\mathcal{N}^+_{s^+,r}$. We now prove our main result.

Theorem 2. *Any non-flat shape can ultimately perform crossing.*

Proof. The proof has two mains stages. First, we describe a construction of points in $\mathbb{R}^2$, and second we apply Lemma 4 to convert it to a crossing configuration for $\mathcal{N}^+_{s^+,r}$ in $\mathbb{Z}^2$.

After defining the setting, at each stage we will first construct the part of the finite crossing configuration where movement vectors (corresponding to arcs of the two firing graphs) do cross each other. Then we will explain how to construct the rest of the configuration in order to connect this crossing part to firing graphs coming from two adjacent borders, and to escape from the crossing part toward the two mirror borders.

Setting. This paragraph is illustrated on Figure 5. Let $\overrightarrow{h}$ be a movement vector of s^+ having the maximal length in terms of Euclidean norm, $h_1 = (0, 0)$, and $h_2 = h_1 + \overrightarrow{h}$. The line (h_1, h_2) cuts the shape s^+ into two parts, s^1 and s^2. We will choose one of these two parts, by considering projections onto the direction orthogonal to $\overrightarrow{h}$. Let $\overrightarrow{v_e}$ be a vector of s^+ whose projection onto the direction orthogonal to $\overrightarrow{h}$ is the longest (in case of equality, take $\overrightarrow{v_e}$ the most orthogonal to $\overrightarrow{h}$). Without loss of generality, let s^2 be the part of s^+ that contains the movement vector $\overrightarrow{v_e}$. We denote $\overrightarrow{s^2_y}$ the projection of $\overrightarrow{v_e}$ onto the direction

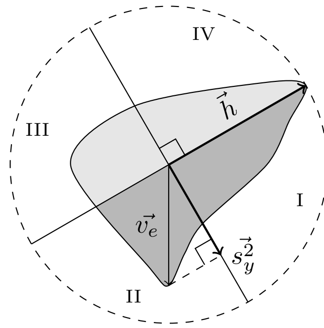


Fig. 5: $\vec{h}$ defines four quadrants pictured with roman numbers.

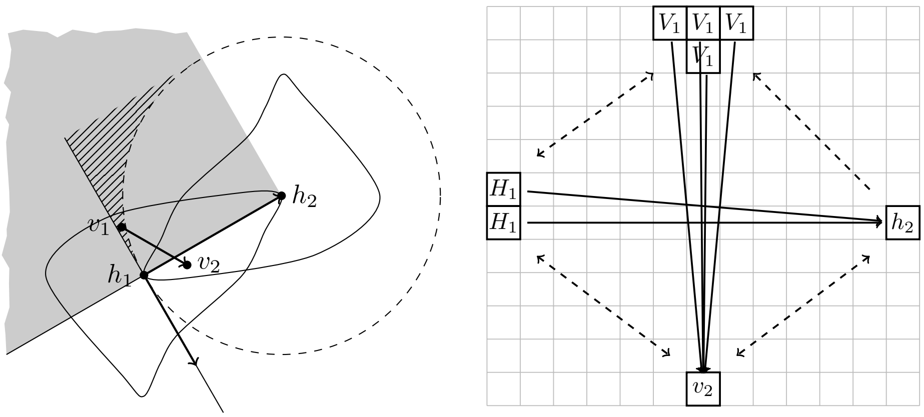
orthogonal to $\vec{h}$. The fact that $\vec{h}$ and $\vec{v_e}$ have some maximality property will be useful in order to come to (resp. escape from) the crossing part towards the West and North (resp. East and South) borders.

Crossing movement vectors (in $\mathbb{R}^2$). We now prove that there always exists a non-null movement vector $\vec{v} \in s^2$, not collinear with $\vec{h}$, that can be placed from v_1 to $v_2 = v_1 + \vec{v}$ in $\mathbb{R}^2$, such that the intersection of line segments $]v_1, v_2[$ and $]h_1, h_2[$ is not empty (loosely speaking, $\vec{h}$ and $\vec{v}$ do cross each other), and most importantly $v_1 \notin s^-(h_2)$, as depicted on Figure 6a. We consider two cases in order to find $\vec{v}$ and v_1.

- If s^+ has a non-flat subshape s' inside the first quadrant, then we take $\vec{v} \in s'$ with strictly positive projections $\vec{v_h}$ and $\vec{v_y}$ onto the direction of $\vec{h}$ and the direction of $\vec{s_y^2}$ (in particular $\vec{v}$ is non-null and not collinear with $\vec{h}$). By maximality of $\vec{h}$, it is always possible to fulfill the requirements, by placing v_1 in the fourth quadrant where we exclude the disk of radius $|\vec{h}|$ centered at h_2 (hatched area on Figure 6a), as close as necessary to h_1. We can for example place v_1 at position $(0,0) - \frac{\vec{v_y}}{2} + \epsilon\vec{h}$ for a small enough $\epsilon \in \mathbb{R}$, $\epsilon > 0$.
- Otherwise s^+ is empty or flat inside the first quadrant, thus $\vec{v_e}$ belongs to the second quadrant, and $s^-(h_2)$ is empty inside the third quadrant relative to h_2 (by symmetry of s^- relative to s^+, darken area on Figure 6a). As a consequence we can for example place v_1 at position $(0,0) + \frac{\vec{h}}{2} - \frac{\vec{v_e}}{2}$, so that $\vec{v} = \vec{v_e}$ and v_1 verify the requirements (s^+ is non-flat therefore $\vec{v_e}$ is non-null and not collinear with $\vec{h}$).

As the shape is non-flat, points h_1, h_2, v_1, v_2 can be converted to small balls $s^+_{h_1}, s^+_{h_2}, s^+_{v_1}, s^+_{v_2}$ (for example by taking a small ball of radius $\epsilon' < \frac{\epsilon}{2}$ around v_1) preserving the neighboring relations among h_1, h_2, v_1, v_2, meaning that every point inside one ball is (or is not) neighbor of every point inside the other ball, in order to later apply Lemma 4.

Coming from two adjacent borders (in $\mathbb{R}^2$). Let us now construct the part of the crossing configuration that connects (in their respective firing graphs)

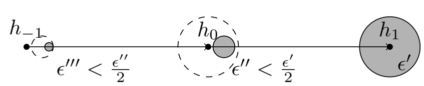


(a) $\overrightarrow{h_1h_2} = \overrightarrow{h}$, the contours of $s^+(h_1)$ and $s^-(h_2)$ are drawn, and the circle of radius $|\overrightarrow{h}|$ centered at h_2 is dashed. We want to find v_1,v_2 with $\overrightarrow{v_1v_2} \in s^2$, such that segments $]v_1v_2[$ and $]h_1, h_2[$ cross each other and $v_1 \notin s^-(h_2)$.

(b) General form of the crossing part of our crossing configuration c. Plain arcs represent the firing graph, and dashed arcs represent possible neighboring relations: we only require that none of the vertices in V_1 has h_2 in their neighborhood ($\mathcal{N}^-(h_2) \cap V_1 = \emptyset$).

Fig. 6: Central part of the crossing configuration (crossing movement vectors in $\mathbb{R}^2$ then $\mathbb{Z}^2$).

two adjacent borders to vertices h_1 and v_1. This can simply be achieved by using the movement vectors $\overrightarrow{h}$ and $\overrightarrow{v_e}$, respectively defining $h_0 = h_1 - \overrightarrow{h}, h_{-1} = h_0 - \overrightarrow{h}, \ldots$ and $v_0 = v_1 - \overrightarrow{h}, v_{-1} = v_0 - \overrightarrow{h}, \ldots$ (see Figure 7), as many times as necessary so that in the horizontal (resp. vertical) firing graph we get a point that is more on the direction of the corresponding adjacent border than any other point. By maximality of these vectors, the coming points towards h_1 (resp. v_1) are not in the neighborhood of the coming points towards v_1 nor v_1 itself (resp. towards h_1 nor h_1 itself). As the shape is non-flat, these points can again be converted to small balls $s^+_{h_0}, s^+_{h_{-1}}, \ldots$ and $s^+_{v_0}, s^+_{v_{-1}}, \ldots$, preserving the neighboring relations. For example as follows:



Escaping toward the two mirror borders (in $\mathbb{R}^2$). Escaping from the crossing part towards the two mirror borders is identical to coming from the two adjacent borders, again using the maximality of movement vectors $\overrightarrow{h}$ and $\overrightarrow{v_e}$. This defines $s^+_{h_3}, \ldots$ and $s^+_{v_3}, \ldots$.

Crossing movement vectors (in $\mathbb{Z}^2$). Let us now explain how Lemma 4 can be used to convert the two finite sets of small balls we have been defining

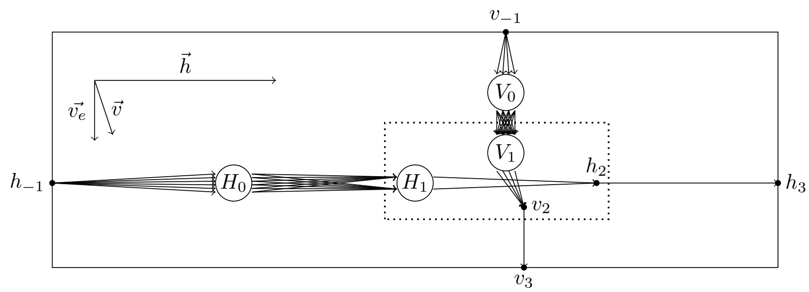


Fig. 7: Global illustration of the crossing configuration. The crossing part (of movement vectors $\overrightarrow{h}$ and $\overrightarrow{v}$) is dotted. To come from two adjacent borders and escape toward the two mirror borders, the horizontal and vertical firing graphs respectively use movement vectors $\overrightarrow{h}$ and $\overrightarrow{v_e}$.

(one set of small balls for the horizontal firing graph, and one set of small balls for the vertical firing graph), to a crossing configuration in $\mathbb{Z}^2$.

We claim that the fact that $v_1 \notin s^-(h_2)$ allows one to construct the crossing part of the crossing configuration as described on Figure 6b, for $\mathcal{N}^+_{s^+,r}$ when r is big enough. Small balls $s^+_{h_1}, s^+_{h_2}, s^+_{v_1}, s^+_{v_2}$ are respectively converted to sets $H_1, \{v_2\}, V_1, \{v_2\}$.

In this construction, we want V_1 to fire v_2, and H_1 to fire h_2, and also $V_1 \cup \{v_2\}$ to not fire H_1 nor h_2, and $H_1 \cup \{h_2\}$ to not fire V_1 nor v_2. Hence we have to ensure for example that $|H_1| > |\{v_2\} \cap \mathcal{N}^-(h_2)|$ and for all $h_1 \in H_1$, $c(h_1) < p - |\mathcal{N}^-(h_1) \cap (V_1 \cup \{v_2\})|$. All these conditions are verified with $|H_1| = 2$, $|V_1| = 4$, $c(h_1) = p-6$ for all $h_1 \in H_1$, $c(h_2) = p-2$, $c(v_1) = p-4$ for all $v_1 \in V_1$, $c(v_2) = p-4$, which can be obtained from Lemma 4 for $r \geq r_1$.

Coming from two adjacent borders (in $\mathbb{Z}^2$). Vertices of H_1 need to receive six grains each. Let us describe the construction in the reverse direction: starting from H_1 backward to a border, in two steps. At step one, we have $s^+_{h_0}$ that will be converted to a set $|H_0| = 6$ such that $c(h_0) = p-1$ for all $h_0 \in H_0$, by Lemma 4. At step two, all the subsequent $h_{-1}, \ldots$ are respectively converted to sets $\{h_{-1}\}, \ldots$ such that $c(h_{-1}) = p-1, \ldots$, until we reach the corresponding adjacent border which defines a vector for the crossing configuration, let say $\overrightarrow{w} \in E_n$. The same construction for the vertical firing graph (with $s^+_{v_0}$ converted to $|V_0| = 4$) gives a vector $\overrightarrow{n} \in E_n$. Let r_2 be the maximal ratio given by Lemma 4 for this part of the construction.

Escaping toward the two mirror borders (in $\mathbb{Z}^2$). Escaping from the crossing part towards the two mirror borders is again identical to coming from the two adjacent borders, with only sets of size one. Let $\overrightarrow{e}, \overrightarrow{s}$ be the corresponding vectors on the two mirror borders, and r_3 be the maximal ratio given by Lemma 4 for this part of the construction.

Conclusion. Recall Remark 3: the crossing configuration may be in any orientation. In the particular case that the constructed crossing has directions $\vec{h}$ and $\vec{v_e}$ pointing towards the corners of the crossing configuration ($\vec{h} = (1,1)$ and $\vec{v_e} = (1,-1)$ for example), then it can easily be made a proper crossing with well defined North, East, South, West vectors by slightly changing the directions near the borders, thanks to the fact that the shape is non-flat.

Note that for simplicity we applied Lemma 4 multiple times, but it can equally be applied once, giving some ratio $r_0 \geq \max\{r_1, r_2, r_3\}$ from which shape s^+ performs crossing: indeed the obtained configuration is finite, stable, and with vectors $\vec{n}, \vec{e}, \vec{s}, \vec{w}$ it transports from two adjacent borders to the two mirror borders, with isolation, *i.e.*, it is a crossing configuration. $\square$

5 Conclusion and perspective

After giving a precise definition of crossing configurations in the abelian sandpile model on $\mathbb{Z}^2$ with uniform neighborhood, we have proven that the corresponding firing graphs can always be chosen to be disjoint. We have seen that this fact has consequences on the impossibility to perform crossing for some neighborhoods with short movement vectors, and that crossing configurations with convex neighborhoods require some involved constructions with firing cells having at least two in-neighbors in the firing graphs. Finally, we proved the main result that any shape can ultimately perform crossing (Theorem 2).

As a consequence of Theorem 2, the conditions on a neighborhood such that it cannot perform crossing cannot be expressed in continuous terms, but are intrinsically linked to the discreteness of neighborhoods. It remains to find such conditions, *i.e.*, to characterize the class of neighborhoods that cannot perform crossing. More generally, what can be said on the set of neighborhoods that cannot perform crossing? It would also be interesting to have an algorithm to decide whether a given neighborhood can perform crossing or not, since the decidability of this question has not yet been established.

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