A Criterion for Robust Stability with Respect to Parametric Uncertainties Modeled by Multiplicative White Noise with Unknown Intensity, with Applications to Stability of Neural Networks

Vasile Dragan, Adrian-Mihail Stoica, Toader Morozan

A Criterion for Robust Stability with Respect to Parametric Uncertainties Modeled by Multiplicative White Noise with Unknown Intensity, with Applications to Stability of Neural Networks

Vasile Dragan[a], Adrian-Mihail Stoica[b], and Toader Morozan[a]

[a]Institute of Mathematics of the Romanian Academy,
P.O. Box 1-764, Ro-70700, Bucharest, Romania
[b]University "Politehnica" of Bucharest,
Faculty of Aerospace Engineering, Bucharest, Romania
{vasile.dragan@imar.ro, adrian.stoica@upb.ro, teoader.morozan@imar.ro}

Abstract. In the present paper a robust stabilization problem of continuous-time linear dynamic systems with Markov jumps and corrupted with multiplicative (state-dependent) white noise perturbations is considered. The robustness analysis is performed with respect to the intensity of the white noises. It is proved that the robustness radius depends on the solution of an algebraic system of coupled Lyapunov matrix equations.

Keywords: Stochastic systems, Markovian jumps, multiplicative white noise, robust stability, Lyapunov operators

1 Introduction

The stochastic systems subject both to Markovian jumps and to multiplicative white noise perturbations received a considerable attention over the last years. Relevant results include the stability of such systems, optimal control and filtering (see e.g. [4], [5], [6], [8] and their references). In the present paper a robust stabilization problem of continuous-time linear dynamic systems with Markov jumps and corrupted with multiplicative (state-dependent) white noise perturbations is considered. The robustness analysis is performed with respect to the intensity of the white noise terms. It is proved that the robustness radius depends on the solution of an algebraic system of coupled Lyapunov matrix equations. The derived results are a generalization of the ones proved in [10] for the case without Markovian jumps. The paper is organized as follows: in the next section the problem statement is presented. The third section includes some preliminary results concerning the Lyapunov operators associated to the considered class of stochastic systems. The main result is presented and proved in Section 4. In the last section the stability radius is determined for two relevant particular cases and a numerical example illustrates the theoretical developments.

2 The problem statement

Consider the system of stochastic linear differential equations:

$$dx(t) = A(\eta_t)x(t)dt + \sum_{l=1}^{r} \mu_l b_l(\eta_t)c_l^T(\eta_t)x(t)dw_l(t) \tag{1}$$

where $\{w_l(t)\}_{t\geq 0}$, $1 \leq l \leq r$, are one-dimensional independent standard Wiener processes defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$; $\{\eta_t\}_{t\geq 0}$ is a homogeneous standard right continuous Markov process defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and taking value in the finite set $\mathbb{N} = \{1, 2, ..., N\}$ and having the transition semigroup $P(t) = e^{Qt}$, $t \geq 0$, where $Q \in \mathbb{R}^{N\times N}$ is a matrix whose elements q_{ij} satisfy the condition

$$\begin{cases} q_{ij} \geq 0 \; if \; i \neq j, \; i, j \in \mathbb{N} \\ \sum\limits_{j=1}^{N} q_{ij} = 0, \forall i \in \mathbb{N}. \end{cases} \tag{2}$$

For more details we refer to [1], [3], [9], [12]. We also assume that $\{\eta_t\}_{t\geq 0}$, $\{w_l(t)\}_{t\geq 0}$, $1 \leq l \leq r$, are independent stochastic processes. In (1) the matrices $A(i) \in \mathbb{R}^{n\times n}$, $b_l(i)$, $c_l(i) \in \mathbb{R}^{n\times 1}$, $1 \leq l \leq r$, $1 \leq i \leq N$ are known, while the scalars $\mu_l \in \mathbb{R}$ are unknown. The system (1) can be regarded as a perturbation of the so called nominal system

$$\dot{x}(t) = A(\eta_t)x(t). \tag{3}$$

The perturbed system (1) emphasizes the fact that the coefficients of the nominal system are affected by parametric uncertainties modeled by state multiplicative white noise perturbations with unknown intensity μ_l. Often when we refer to the perturbed system (1) we shall say that it corresponds to the vector of parameters $\mu = (\mu_1, \mu_2, ..., \mu_r)$. Assuming that the nominal system (3) is exponentially stable in mean square (ESMS) we want to find necessary and sufficient conditions which will be satisfied by the parameters $\mu_l, 1 \leq l \leq r$ such that the perturbed system (1) to be also ESMS. In the special case $\mathbb{N} = \{1\}$ (no Markov jumps) the conditions derived in this note recover those derived in [10]. The concept of exponential stability in mean square of the linear stochastic systems of type (1) and (3) may be found in [2] and [8], respectively.

3 Some preliminaries

Let $\mathcal{S}_n^N = \mathcal{S}_n \otimes \mathcal{S}_n \otimes ... \otimes \mathcal{S}_n$, where $\mathcal{S}_n \subset \mathbb{R}^{n\times n}$ is the subspace of symmetric matrices. Let $\mathcal{S}_{n+}^N$ be the convex cone defined by $\mathcal{S}_{n+}^N = \{\mathbf{X} = (X(1), X(2), ..., X(N)) \in \mathcal{S}_n^N | X(i) \geq 0, 1 \leq i \leq N\}$. Here, $X(i) \geq 0$ means that $X(i)$ is positive semidefinite, $\mathcal{S}_{n+}^N$ is a closed convex cone with non empty interior. Its interior is $Int\mathcal{S}_{n+}^N = \{\mathbf{X} = (X(1), X(2), ..., X(N)) \in \mathcal{S}_{n+}^N | X(i) > 0, 1 \leq i \leq N\}$. Applying Theorem

3.3.2 and Theorem 3.3.3 from [8] in the case of system (1) one obtains the following result.

Proposition 3.1 *The following are equivalent*

(i) The system (4) is ESMS;

(ii) For any $\mathbf{H} = (H(1), H(2), ..., H(N)) \in Int\mathcal{S}_{n+}^N$ *there exists* $\mathbf{X} = (X(1), X(2), ..., X(N)) \in Int\mathcal{S}_{n+}^N$ *solving the following equation on* $\mathcal{S}_n^N$

$$\begin{aligned} &A^T(i)X(i) + X(i)A(i) + \sum_{j=1}^{N} q_{ij}X(j) + \sum_{l=1}^{r} \mu_l^2 c_l(i) b_l^T(i) X(i) b_l(i) c_l^T(i) \\ &+H(i) = 0; 1 \le i \le N; \end{aligned} \tag{4}$$

(iii) For any $\mathbf{H} \in Int\mathcal{S}_{n+}^N$*, there exists* $\mathbf{Y} = (Y(1), y(2), ..., Y(N)) \in Int\mathcal{S}_{n+}^N$ *solving the following equation on* $\mathcal{S}_n^N$

$$\begin{aligned} &A(i)Y(i) + Y(i)A^T(i) + \sum_{j=1}^{N} q_{ji}Y(j) + \sum_{l=1}^{r} \mu_l^2 b_l(i) c_l^T(i) Y(i) c_l(i) b_l^T(i) \\ &+H(i) = 0, 1 \le i \le N; \end{aligned} \tag{5}$$

(iv) There exists $\mathbf{X} = (X(1), ..., X(N)) \in Int\mathcal{S}_{n+}^N$ *satisfying the following system of LMIs*

$$A^T(i)X(i) + X(i)A(i) + \sum_{j=1}^{N} q_{ij}X(j) + \sum_{l=1}^{r} \mu_l^2 c_l(i) b_l^T(i) X(i) b_l(i) c_l^T(i) < 0,$$

where $1 \le i \le N$.

Then one associates the following Lyapunov operators to the nominal system (3) $\mathcal{L} : \mathcal{S}_n^N \to \mathcal{S}_n^N$, $\mathfrak{L} : \mathcal{S}_n^N \to \mathcal{S}_n^N$ defined by

$$\mathcal{L}[\mathbf{X}](i) = A^T(i)X(i) + X(i)A(i) + \sum_{j=1}^{N} q_{ij}X(j), 1 \le i \le N, \tag{6}$$

and

$$\mathfrak{L}[\mathbf{X}](i) = A(i)X(i) + X(i)A^T(i) + \sum_{j=1}^{N} q_{ji}X(j), 1 \le i \le N \tag{7}$$

for all $\mathbf{X} = (X(1), ..., X(N)) \in \mathcal{S}_n^N$. Is is easy to check that $\mathcal{L}$ is the adjoint operator of $\mathfrak{L}$ with respect to the usual inner product on $\mathcal{S}_n^N$:

$$< \mathbf{X}, \mathbf{Y} >= \sum_{j=1}^{N} Tr[X(i)Y(i)]. \tag{8}$$

Invoking Proposition 3.20 and Theorem 3.21 from [2] (see also Theorem 3.2.2 and Theorem 3.2.4 from [8]) one deduces that the nominal system (3) is ESMS if and only if the eigenvalues of the linear operator $\mathcal{L}$ are located in the half plane $\mathbb{C}_-$. This allows us to obtain the following result.

Corollary 3.2. *If the nominal system (3) is ESMS then for each* $\mathbf{H} \in \mathcal{S}_n^N$ *the equations*

$$\mathcal{L}[\mathbf{X}] + \mathbf{H} = 0 \tag{9}$$

and

$$\mathfrak{L}[\mathbf{Y}] + \mathbf{H} = 0 \tag{10}$$

have unique solutions given by $\mathbf{X} = -\mathcal{L}^{-1}[\mathbf{H}] = (-\mathcal{L}^{-1}[\mathbf{H}](1), ..., -\mathcal{L}^{-1}[\mathbf{H}](N)) \in \mathcal{S}_n^N$ *and* $\mathbf{Y} = -\mathfrak{L}^{-1}[\mathbf{H}] = (-\mathfrak{L}^{-1}[\mathbf{H}](1), ..., -\mathfrak{L}[\mathbf{H}](N)) \in \mathcal{S}_n^N$*, respectively. If* $\mathbf{H} \in \mathcal{S}_{n+}^N$ *then* $\mathbf{X} \in \mathcal{S}_{n+}^N$, $\mathbf{Y} \in \mathcal{S}_{n+}^N$*. Moreover, if* $\mathbf{H} \in Int\mathcal{S}_{n+}^N$ *then the unique solutions of (9) and (10), respectively are in* $Int\mathcal{S}_{n+}^N$ *i.e.*

$$-\mathcal{L}^{-1}[\mathbf{H}](i) > 0 \tag{11}$$

and

$$-\mathfrak{L}^{-1}[\mathbf{H}](i) > 0 \tag{12}$$

for all $1 \leq i \leq N$.

Further, let us consider the ordered space $(\mathbb{R}^d, \mathbb{R}_+^d)$ where the order relation is induced by the convex cone $\mathbb{R}_+^d = \{x = (x_1, x_2, ..., x_d)^T \in \mathbb{R}^d | x_i \geq 0, 1 \leq i \leq d\}$. The interior $Int\mathbb{R}_+^d$ of the convex cone $\mathbb{R}_+^d$ consists of the set $Int\mathbb{R}_+^d = \{x = (x_1, x_2, ..., x_d)^T \in \mathbb{R}_+^d | x_i > 0, 1 \leq i \leq d\}$. If $D = (d_{ij})_{ij} \in \mathbb{R}^{d\times d}$ is the matrix of a linear operator $\mathcal{D} : \mathbb{R}^d \to \mathbb{R}^d$ then $\mathcal{D}\mathbb{R}_+^d \subset \mathbb{R}_+^d$ if and only if $d_{ij} \geq 0$, $1 \leq i, j \leq d$. In this case D will be called positive matrix. Applying Theorems 2.6 and 2.7 from [7] in the special case of the ordered linear space $(\mathbb{R}^d, \mathbb{R}_+^d)$ one obtains the following result.

Proposition 3.3. *For a positive matrix* $D \in \mathbb{R}^{d\times d}$ *the following are equivalent:*

(i) $\rho(D) < 1$, $\rho(\cdot)$ *being the spectral radius;*

(ii) There exists $\psi \in Int\mathbb{R}_+^d$ *such that the equation*

$$(I_d - D)\zeta = \psi \tag{13}$$

has a solution $\zeta \in Int\mathbb{R}_+^d$.

4 Main results

The equivalence $(i) \leftrightarrow (iv)$ from Proposition 3.1 allows us to deduce that if the perturbed system (1) is ESMS for a value $\mu = (\mu_1, \mu_2, ..., \mu_r)$ of the intensities of the white noises, then this system is ESMS for every value $\mu^{'} = (\mu_1^{'}, ..., \mu_r^{'})$ which are satisfying $|\mu_l^{'}| \leq |\mu_l|$ for all $1 \leq l \leq r$. In this section we shall derive a set of necessary and sufficient conditions which guarantee the exponential stability in mean square of a perturbed system (1) corresponding to a set of unknown vector of intensities $\mu = (\mu_1, ..., \mu_r)$. Using Proposition 3.1 one notices that the system (1) is ESMS if and only if for any $\mathbf{H} \in Int\mathcal{S}_{n+}^N$ the equation (4) has a solution $\mathbf{X} \in Int\mathcal{S}_{n+}^N$. Using (6) we may rewrite (4) in the form

$$\mathcal{L}[\mathbf{X}] + \tilde{\mathbf{H}} = 0 \tag{14}$$

where $\tilde{\mathbf{H}} = (\tilde{H}(1), ..., \tilde{H}(N))$,

$$\tilde{H}(i) = \sum_{l=1}^{r} \mu_l^2 b_l^T(i) X(i) b_l(i) c_l(i) c_l^T(i) + H(i), 1 \le i \le N. \tag{15}$$

Further, we rewrite $\tilde{\mathbf{H}}$ in the form:

$$\tilde{\mathbf{H}} = \sum_{l=1}^{r} \sum_{j=1}^{N} \mu_l^2 b_l^T(j) X(j) b_l(j) \Xi_{lj} + \mathbf{H} \tag{16}$$

where $\Xi_{lj} = (\Xi_{lj}(1), ..., \Xi_{lj}(N)) \in \mathcal{S}_{n+}^N$ with

$$\Xi_{lj}(i) = \begin{cases} c_l(j) c_l^T(j), \ if \ i = j \\ 0, \ \ \text{otherwise}. \end{cases} \tag{17}$$

Since the nominal system (3) is necessarily ESMS if the perturbed system (1) is ESMS, we deduce via Corollary 3.2 and (16) that the solution of the equation (14) satisfies

$$\mathbf{X} = -\sum_{l=1}^{r} \sum_{j=1}^{N} b_l^T(j) X(j) b_l(j) \mu_l^2 \mathcal{L}^{-1}[\Xi_{lj}] - \mathcal{L}^{-1}[\mathbf{H}].$$

The i^{th} component of this solution is

$$X(i) = -\sum_{l=1}^{r} \sum_{j=1}^{N} b_l^T(j) X(j) b_l(j) \mu_l^2 \mathcal{L}^{-1}[\Xi_{lj}](i) - \mathcal{L}^{-1}[\mathbf{H}](i), 1 \le i \le N. \tag{18}$$

Multiplying (18) on the left by $b_k^T(i)$ and on the right by $b_k(i)$, one obtains

$$b_k^T(i) X(i) b_k(i) = \sum_{l=1}^{r} \sum_{j=1}^{N} b_l^T(j) X(j) b_l(j) m_{ki,lj} + \nu_{ki} \tag{19}$$

$1 \le k \le r$, $1 \le i \le N$, where

$$m_{ki,lj} = -\mu_l^2 b_k^T(i) \mathcal{L}^{-1}[\Xi_{lj}](i) b_k(i) \tag{20}$$

and

$$\nu_{ki} = -b_k^T(i) \mathcal{L}^{-1}[\mathbf{H}](i) b_k(i). \tag{21}$$

One sees that (19) is a system of rN scalars equations with rN scalar unknowns. Based on the fact that for each integer $\alpha \in \{1, 2, ..., rN\}$ there exists a unique pair of natural numbers $(k,i) \in \{1,2,...,r\} \times \{1,2,...,N\}$ such that $\alpha = (k-1)N+i$ we may write (19) as an equation of the form (13) on the space $\mathbb{R}^{rN}$. To this end we set $\zeta = (\zeta_1, \zeta_2, ..., \zeta_{rN})^T$, $\Psi = (\Psi_1, \Psi_2, ..., \Psi_{rN})^T$, $D = (d_{\alpha\beta})_{1\le\alpha,\beta\le rN}$,

$$\zeta_\beta = b_l^T(j)X(j)b_l(j) \; if \; (l-1)N+j=\beta, \tag{22}$$

$$\Psi_\alpha = \nu_{ki} = -b_k^T(i)\mathcal{L}^{-1}[\mathbf{H}](i)b_k(i) \; if \; (k-1)N+i=\alpha \tag{23}$$

$$d_{\alpha\beta} = m_{ki,lj} = -\mu_l^2 b_k^T(i)\mathcal{L}^{-1}[\Xi_{lj}](i)b_k(i) \tag{24}$$

if $(k-1)N+i=\alpha$ and $(l-1)N+j=\beta$. With these notations (19) may be written in a compact form:

$$(I-D)\zeta = \Psi. \tag{25}$$

Since the matrix D defined by (24) depend upon the unknown parameters μ_l, for each perturbed system of type (1) one may associate a matrix $D = D(\mu)$ as before. Now we are in a position to state and proof the following result.

Theorem 4.1 *Assume* $b_k(i) \neq 0$, $1 \le k \le r$, $1 \le i \le N$. *Under this condition the following are equivalent:*

(i) The perturbed system (1) corresponding to the set of parameters $\mu = (\mu_1, ..., \mu_r)$ *is ESMS;*

(ii) The nominal system (3) is ESMS and the matrix $D(\mu)$ *associated to the perturbed system (1) satisfies* $\rho(D(\mu)) < 1$.

Proof $(i) \Rightarrow (ii)$ Let $\mathbf{H} \in Int\mathcal{S}_{n+}^N$ be arbitrary but fixed. If the perturbed system (1) is ESMS, then the nominal system (3) is also ESMS. Hence, the equation (14)-(15) has a solution $\mathbf{X} = (X(1), ..., X(N))$ with $X(i) > 0$, $1 \le i \le N$. Consider $\zeta, \Psi \in \mathbb{R}^{rN}$ defined via (22) and (23). Since $b_k(i) \neq 0$, for all $(k,i) \in \{1,2,...,r\} \times \{1,2,...,N\}$ one deduces that $\zeta_\alpha > 0$ and $\Psi_\alpha > 0$, $1 \le \alpha \le rN$. Further, one associates the matrix D whose elements are computed via (24). One may check that $d_{\alpha\beta} \ge 0$, for all $1 \le \alpha, \beta \le rN$. So, it follows that the equation (25) associated to the perturbed system (1) satisfies the conditions from Proposition 3.3 (ii). Hence, $\rho(D) < 1$, this shows that the assertion (ii) from the statement holds if (i) is satisfied.

Now the implication $(ii) \Rightarrow (i)$ will be proved. Let $\mathbf{H} \in Int\mathcal{S}_{n+}^N$ be arbitrary. It will be shown that the corresponding equation (4) has a solution $\mathbf{X} \in Int\mathcal{S}_{n+}^N$. One notices that if the nominal system (3) is ESMS, then $\Psi \in Int\mathbb{R}^{rN}$ and the matrix $D \in \mathbb{R}^{rN \times rN}$ are well defined via (23) and (24), respectively. If $\rho(D) < 1$ then, based on the implication $(i) \Rightarrow (ii)$ from Proposition 3.3 it follows (25) has a unique solution $\zeta \in Int\mathbb{R}_+^{rN}$. Based on the components of the vectors Ψ, ζ and of the matrix D one may define

$$\tilde{\Psi}_{ki} = \Psi_\alpha, \tilde{\zeta}_{lj} = \zeta_\beta, \tilde{d}_{ki,lj} = d_{\alpha\beta} \tag{26}$$

if $(k,i),(l,j) \in \{1,2,...,r\} \times \{1,2,...,N\}$, $(k-1)N+i=\alpha, (l-1)N+j=\beta$. It is easy to see that $\tilde{\Psi}_{ki} = \Psi_{ki}$, $\tilde{d}_{ki,lj} = m_{ki,lj}$. With these notations one obtains the following version of the equation (25)

$$\tilde{\zeta}_{ki} = -\sum_{l=1}^{r}\sum_{j=1}^{N}\mu_l^2 b_k^T(i)\mathcal{L}^{-1}[\Xi_{lj}](i)b_k(i)\tilde{\zeta}_{lj} - b_k^T(i)\mathcal{L}^{-1}[\mathbf{H}](i)b_k(i) \tag{27}$$

$\forall (k,i) \in \{1,2,...,r\} \times \{1,2,...,N\}$. Defining

$$X(i) = -\sum_{l=1}^{r}\sum_{j=1}^{N}\mu_l^2\tilde{\zeta}_{lj}\mathcal{L}^{-1}[\Xi_{lj}](i) - \mathcal{L}^{-1}[\mathbf{H}](i), 1 \le i \le N. \tag{28}$$

and setting $\mathbf{X} = (X(1),...,X(N))$, it results that $\mathbf{X} \in Int\mathcal{S}_{n+}^N$ since $-\mathcal{L}^{-1}[\Xi_{lj}](i) \ge 0$, $-\mathcal{L}^{-1}[\mathbf{H}](i) > 0$ and $\tilde{\zeta}_{lj} > 0$ for all $(l,j,i) \in \{1,2,...,r\} \times \{1,2,...,N\} \times \{1,2,...,N\}$. From (28) it follows

$$\mathcal{L}[\mathbf{X}] + \sum_{l=1}^{r}\sum_{j=1}^{N}\mu_l^2\tilde{\zeta}_{lj}\Xi_{lj} + \mathbf{H} = 0. \tag{29}$$

Based on (17) it results that $\Xi_{lj}(i) = 0$ if $i \ne j$, obtaining thus the following componentwise version of (29)

$$\mathcal{L}[\mathbf{X}](i) + \sum_{l=1}^{r}\mu_l^2\tilde{\zeta}_{li}c_l(i)c_l^T(i) + H(i) = 0, 1 \le i \le N. \tag{30}$$

From (28) with (27) it results that $\tilde{\zeta}_{ki} = b_k^T(i)X(i)b_k(i)$ for all $1 \le k \le r$, $1 \le i \le N$. Therefore one may rewrite (30) in the form

$$\mathcal{L}[\mathbf{X}](i) + \sum_{l=1}^{r}\mu_l^2 c_l(i)b_l^T(i)X(i)b_l(i)c_l^T(i) + H(i) = 0 \tag{31}$$

which is just (4). Thus the proof is complete.

Remark 4.1 The result proved in Theorem 4.1 shows that, in order to decide if the perturbed system (1) corresponding to a vector of unknown parameters $\mu_l \in [-|\tilde{\mu}_l|, |\tilde{\mu}_l|]$, $1 \le l \le r$ is ESMS, we have to check if the spectral radius of the matrix $D = D(\tilde{\mu})$ (associated via (24) to the parameters $\tilde{\mu}_l$) is less then 1. One notices that (24) may be rewritten in the form

$$d_{\alpha\beta} = \tilde{\mu}_l^2 b_k^T(i)Z_{lj}(i)b_k(i) \tag{32}$$

where $(k,i),(l,j) \in \{1,2,...,r\} \times \{1,2,...,N\}$ are such that $(k-1)N + i = \alpha$ and $(l-1)N + j = \beta$, $\mathbf{Z}_{lj} = (Z_{lj}(1),...,Z_{lj}(N))$ being the unique solution of the equation

$$A^T(i)Z_{lj}(i) + Z_{lj}(i)A(i) + \sum_{\iota=1}^{N}q_{i\iota}Z_{lj}(\iota) + \Xi_{lj}(i) = 0 \tag{33}$$

$1 \le i \le N$, with Ξ_{lj} defined in (17).

According with [10] one introduces the following definition.

Definition 4.1 The vector of noise intensities $\mu^0 = (\mu_1^0, \mu_2^0, ..., \mu_r^0)$ is called critical for the system (1) if the system (1) corresponding to the noise intensities $\varepsilon\mu^0$ is ESMS if $0 < \varepsilon < 1$ and it is not ESMS if $\varepsilon \ge 1$.

Remark 4.2 Denote $D(\mu)$ the matrix D corresponding to the vector $\mu = (\mu_1, ..., \mu_r)$ of the noise intensities. Based on (24) it follows that $D(\varepsilon\mu^0) = \varepsilon^2 D(\mu^0)$. Since the spectral radius of a positive matrix is an eigenvalue of that matrix one may infer that

$$\rho[D(\varepsilon\mu^0)] = \varepsilon^2 \rho[D(\mu^0)].$$

If μ^0 is a critical vector of noise intensities then one obtains from Theorem 4.1 that $\rho[D(\varepsilon\mu^0)] < 1$ for all $0 < \varepsilon < 1$ obtaining thus that $\rho[D(\mu^0)] < \frac{1}{\varepsilon^2}$, $0 < \varepsilon < 1$. So, we deduce that $\rho[D(\mu^0)] \le 1$. Since the perturbed system (1) corresponding to the vector μ^0 is not ESMS, one concludes via Theorem 4.1, that $\rho[D(\mu^0)] = 1$. Hence, the vector $\mu^0 = (\mu_1^0, ..., \mu_r^0)$ is a solution of the equation $det(I_{rN} - D(\mu)) = 0$.

In the space $\mathbb{R}^r$ of the vector $\mu = (\mu_1, ..., \mu_r)$ the critical vectors of noise intensities μ^0 are included in the boundary of the stability region.

5 Several special cases

The first special case analyzed here is $r = 1$ and $N \ge 2$. In this case the system (1) becomes

$$dx(t) = A(\eta_t)x(t)dt + \mu b(\eta_t)c^T(\eta_t)x(t)dw_1(t). \tag{34}$$

The matrix D associated to the system (34) is $D = \mu^2 D_1$ where

$$D_1 = \begin{pmatrix} b^T(1)Z_1(1)b(1) & b^T(1)Z_2(1)b(1) & ... & b^T(1)Z_N(1)b(1) \\ b^T(2)Z_1(2)b(2) & b^T(2)Z_2(2)b(2) & ... & b^T(2)Z_N(2)b(2) \\ ... & ... & ... & ... \\ b^T(N)Z_1(N)b(N) & b^T(N)Z_2(N)b(N) & ... & b^T(N)Z_N(N)b(N) \end{pmatrix} \tag{35}$$

for each $1 \le j \le N$, $(Z_j(1), ..., Z_j(N))$ is the unique solution of the following equation on $\mathcal{S}_n^N$

$$A^T(i)Z_j(i) + Z_j(i)A(i) + \sum_{\iota=1}^{N} q_{i\iota} Z_j(\iota) + \Xi_j(i) = 0 \tag{36}$$

$\Xi_j(i) = 0$ if $i \ne j$ and $\Xi_j(i) = c(j)c^T(j)$ if $i = j$.

In the special case of the system (34), the Theorem 4.1 yields to the following result.

Corollary 5.1 *If $b(i) \ne 0$, $\forall 1 \le i \le N$ the following are equivalent:*

(i) The perturbed system (34) is ESMS;

(ii) The nominal system (3) is ESMS and the parameter μ satisfies the condition $\mu^2 < \frac{1}{\rho[D_1]}$.

Remark 5.1 The previous Corollary shows that the exponential stability in mean square of the nominal system (3) is preserved for the perturbed system (34) if and only if the unknown parameter μ lies in the interval $(-\rho^{\frac{-1}{2}}[D_1], \rho^{\frac{-1}{2}}[D_1])$, which is the stability region in the case of perturbed system (34).

The second special case discussed here is $r \geq 1$, $N = 1$. Now, the system (1) becomes

$$dx(t) = Ax(t)dt + \sum_{l=1}^{r} \mu_l b_l c_l^T x(t) dw_l(t). \tag{37}$$

The matrix D associated to the system (37) via (24) is

$$D = \hat{D} diag(\mu_1^2, ..., \mu_r^2), \tag{38}$$

where

$$\hat{D} = \begin{pmatrix} b_1^T Z_1 b_1 & b_1^T Z_2 b_1 & ... & b_1^T Z_r b_1 \\ b_2^T Z_1 b_2 & b_2^T Z_2 b_2 & ... & b_2^T Z_r b_2 \\ ... & ... & ... & ... \\ b_r^T Z_1 b_r & b_r^T Z_2 b_r & ... & b_r^T Z_r b_r \end{pmatrix} \tag{39}$$

for each $1 \leq l \leq r$, Z_l is the unique solution of the Lyapunov equation

$$A^T Z_l + Z_l A + c_l c_l^T = 0. \tag{40}$$

The result proved in Theorem 4.1 yields the next result.

Corollary 5.2 *The following are equivalent*

(i) The perturbed system (37) is ESMS for any value of the unknown parameters $\mu_l \in (-|\tilde{\mu}_l|; |\tilde{\mu}_l|)$, $1 \leq l \leq r$.

(ii) A is a Hurwitz matrix and $\rho[D] < 1$, D being defined in (38)-(39) with μ_l replaced by $\tilde{\mu}_l, 1 \leq l \leq r$.

Remark 5.2 a) The result stated in Corollary 5.2 is just the main result derived in [10]. Its discrete-time version may be found in [11].

b) Condition of the form $b_l \neq 0, 1 \leq l \leq r$ (as it is imposed in the general case in Theorem 4.1) is redundant in the case of system (37) because, if $b_{l_0} = 0$ for some l_0, it follows that the noise $w_{l_0}(t)$ does not affect the perturbed system.

In order to illustrate the above theoretical results one considers the dynamics of Hopfield neural network of form

$$\dot{v}_i(t) = a_i v_i(t) + \sum_{j=1}^{n} b_{ij} g_j \left(v_j(t)\right) + c_i, i = 1, ..., N$$

$a_i < 0$ and the *activation functions* $g_i(\cdot)$ are strictly increasing.Then its approximation around an equilibrium point v^0 is

$$\dot{x}(t) = Ax(t) + Bf(x(t))$$

where $x(t) = v(t) - v^0$, $f(x) = g(x + v^0) - g(v^0)$ and where $A = diag(a_1, ..., a_n)$ the elements of $f(\cdot)$ being *sector-type* nonlinearities satisfying $f_k(x_k)\,(f_k(x_k) - \mu_k x_k) \leq 0$, $k = 1, ..., n$. Then for the above system associate the linear approximation

$$\begin{array}{ll} dx(t) & = Ax(t)dt + \mu \sum_{l=1}^{r} b_l c_l^T x(t) dw_l(t),\; n = 3,\; r = 2 \\ A & = diag\,(-0.5,\, -0.5,\, -0.5) \\ b_1 & = [\,0.5,\, 1,\, 1\,]^T,\; c_1 = [\,1,\, 2,\, 1\,]^T, \\ b_2 & = [\,1,\, 0.25,\, -1\,]^T,\; c_2 = [\,0.25,\, -1\,,\, 1\,]^T \end{array}$$

Applying Corollary 5.2 it results that the above system is ESMS for all $\mu \in [-0.2857; 0.2857]$.

Acknowledgement

This paper has been partially supported by MEN-UEFISCDI, Program Partnerships, Projects PN-II-PT-PCCA- 2013-4-1349 and PN-II-PT-PCCA- 2011-3.1-1560.

References

1. Chung,K.L.: Markov chains with stationary transition probabilities, Springer-Verlag, Berlin (1967)
2. Costa, L.V., Fragoso, M.D., Todorov, M.G.: Continuous-time Markov jump linear systems, Springer, London (2013)
3. Doob, J.L.: Stochastic Processes , Wiley, New York, (1967)
4. Huang, Y., Zhang, W., Feng, G.: Infinite horizon H_2/H_∞ control for stochastic systems with Markovian jumps. Automatica vol. 4 (2008)
5. Mariton, M.: Jump Linear Systems in Automatic Control. Marcel Dekker, New York (1990)
6. Li, X., Zhou, X.Y., Rami, M.A.: Indefinite stochastic linear quadratic control with Markovian jumps in indefinite time horizon. Journal of Global Optimization, vol. 27 (2003).
7. Dragan, V., Morozan, T., Stoica, A.-M.: Mathematical Methods in Robust Control of Discrete-time Linear Stochastic Systems. Springer, New-York (2010)
8. Dragan, V., Morozan, T., Stoica, A.-M.: Mathematical Methods in Robust Control of Linear Stochastic Systems Springer, New-York, Second Edition (2013)
9. Friedman, A.: Stochastic Differential Equations and Applications, vol. I, Academic, New York (1975)
10. Levit, M.V., Yakubovich, V.A.: Algebraic criterion for stochastic stability of linear systems with parametric action of white noise type. Applied Math. and Mech., no. 1 (1972)
11. Morozan, T.: Necessary and sufficient conditions of stability of stochastic discrete systems. Rev. Roum. Math. Pures et Appl., no.2 (1973)
12. Oksendal, B.: Stochastic Differential Equations, Springer, Berlin (1998)