Interference Alignment in Reduced-Rank MIMO Networks with Application to Dynamic TDD

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***Abstract*—In Dynamic Time Division Duplex (DynTDD), downlink/uplink (DL/UL) slot allocation is adaptive with traffic load. DynTDD systems have received a lot of attention for 5th generation (5G) mobile communication systems, as the spectrum efficiency of wireless communication networks is improved by the flexible and dynamic duplex operation. However when using DynTDD, a different DL/UL slot configuration is likely to be selected by neighboring cells, leading to Cross Link Interference (CLI) between the Base Stations (BS), which is known as BS-to-BS or DL-to-UL interference, and between User Equipment (UE) which is known as UE-to-UE or UL-to-DL interference. Rank deficient channels are frequently encountered in Multi-Input Multi-Output (MIMO) networks, due to poor scattering and keyhole effects, or when using Massive MIMO and moving to mmWave. While the implications of rank deficient channels are well understood for the single user (SU) point to point setting, less is known for interference networks. In this paper, we extend a MIMO Interference Alignment (IA) feasibility investigation framework to rank deficient channels and we investigate the IA feasibility in DynTDD considering rank deficient MIMO interfering channels, by establishing the simultaneously necessary and sufficient conditions. These conditions allow us to evaluate the Degrees-of-Freedom (DoF) of centralized designs more precisely compared to the loose proper conditions. We then compare the achievable DoF of centralized designs with those of various distributed designs, allowing us to get a better idea of the DoF price to pay for distributedness (but not accounting for the gain in information exchange reduction that distributed designs permit).**

***Index Terms*—Dynamic TDD, MIMO, rank deficient, interference alignment, Degree of Freedom**

I. INTRODUTION

Dynamic Time Division Duplexing (DynTDD) is one promising way to improve the spectrum efficiency of the wireless communication networks since flexible traffic adaptation can be achieved by dynamically changing uplink (UL) or downlink (DL) transmission direction. DynTDD performance has been analyzed in the literature. [1] investigates the impact of synchronous DynTDD on the performance of the DL/UL in dense small cell networks (SCNs). The results show that DynTDD outperforms the static TDD in terms of average total Time Resource Allocation (TRA), and DL and UL Area Spectral Efficiency (ASE). However, DynTDD also brings some new challenges because of the introduction of cross-link interference (CLI), including DL-to-UL interference (e.g., gNB-to-gNB interference) and UL-to-DL interference (UE-to-UE interference). In this scope many techniques have been proposed for the cross-link interference mitigation. Some authors propose solutions based on an optimization problem, such as Mean Square Error (MSE) minimization with the constraint of maximum transmit power for the DL BS and UL UE in [2], and the minimization of the total transmit sum power while satisfying a minimum signal-to-interference-plus-noise ratio (SINR) threshold for every UE in [3].

The feasibility conditions of IA have been analyzed in [4]–[10] . [11] also mathematically characterizes the achievable Degrees-of-Freedom (DoF) of their proposed DIA technique for a given number of antennas at BS/MS. In [4] the authors show that for the MIMO Interfering Broadcast Channel (IBC) where each user has one desired data stream, a proper system is feasible. For the symmetric (which we call here uniform) MIMO Interfering Broadcast Channel (IBC), they provide a proper but infeasible region of antenna configurations by analyzing the difference between the necessary conditions and the sufficient conditions of linear IA feasibility. In [4] the authors analyze the feasibility of linear IA for the MIMO IBC with constant coefficients. They pose and prove the necessary conditions of linear IA feasibility for general MIMO IBC. Except for the proper condition, they find another necessary condition to ensure a kind of irreducible interference to be eliminated. Then they prove the necessary and sufficient conditions for a special class of MIMO IBC, where the numbers of antennas are divisible by the number of data streams per user. [10] established a necessary and sufficient condition on IA feasibility for the (full rank) MIMO Interfering Broadcast Multiple Access Channel (IBMAC), which characterizes the optimal sum DoF for various practical network configurations. [12] addresses (centralized) attainable (only) DoF for general interference networks with general channel rank conditions. The multiple antennas give each node a certain zero-forcing (ZF) budget that for a given DoF distribution needs to be coordinated between all nodes to handle all interference. [13] provides also an approach to find the spatial filter matrices that offer the desired DoF scheduling and reduce the unwanted interference signal strength to close to zero (rather than absolute zero). In [14]

ISBN 978-3-903176-49-2 © 2022 IFIP

IA is explored in a cognitive radio interference channel with only a single primary and a single secondary user, but involving possibly rank deficient channels. In the two user case, IA solutions can be derived analytically.

Contributions: The work reported in this paper goes beyond our study in [15] considering the rank deficient MIMO IBMAC Interference Channel (IC). We start in this paper by reporting the proper conditions for IA feasibility considering a centralized design. We refer the reader to [15, eq(6)] as well as the four points mentioned before this equation, for a discussion on centralized design and various distributed designs, some partitioned between transmitters (Tx) and receivers (Rx), some based on one-sided zero-forcing (ZF). In [15], we provide necessary and sufficient conditions (i.e. tight feasibility conditions) for the distributed designs, but to evaluate their DoF loss, we only compare to the proper conditions for centralized designs, knowing that those conditions are optimistic. Hence in this paper we analyze necessary and sufficient conditions for centralized designs, in order to get a more precise idea of the loss in DoF of distributed vs. centralized designs.

We extend the well known full rank Jacobian matrix condition for the full MIMO channel rank Interference Channel (IC) system [4]–[6], [10] to cover the more general scenario of rank deficient MIMO IC system. The original necessary and sufficient DoF feasibility conditions are based on a novel simplified parameterization of rank deficient MIMO channels (simplified for the purpose of IA feasibility analysis as in [4]–[6]). At the end we consider a numerical scenario for which we provide a comparative table of the achievable DoF considering the different conditions in [15] and the necessary and sufficient condition given by Theorem 4. We conclude that distributed methods can be optimal for low rank scenarios, and that they are suboptimal compared to (unrealistic?) centralized designs, but not by as much as the proper conditions would have led us to believe [10].

II. Dynamic TDD System Model

We consider a MIMO system with two cells, one operating in DL and the other one in UL. Each cell has one BS of M antennas, with K_{ul} and K_{dl} interfering/interfered users in the UL and DL cell respectively. The kth DL UE and the lth UL UE have $N_{dl,k}$ and $N_{ul,l}$ antennas respectively. This scenario brings the two types of interference, the BS-to-BS interference, and the UE-to-UE interference between the UEs that are particularly on the edge of the two cells as shown in Fig 1. The channel between the lth user in the UL cell and the kth user in the DL cell is denoted as $\boldsymbol{H}_{k,l} \in \mathbb{C}^{N_{dl,k} \times N_{ul,l}}$ with $k \in [1, ..., K_{dl}]$ and $l \in [1, ..., K_{ul}]$. Denote $d_{dl,k}$ and $d_{ul,l}$ as the number of data stream from the DL BS to the kth DL UE and from the lth UL UE to the UL BS respectively.

We denote the rank of the UE-to-UE interference channel (IC) as $r_{k,l}$. We have $r_{k,l}$ distinguishable significant paths contribute to $\boldsymbol{H}_{k,l}$. Then we can factorize $\boldsymbol{H}_{k,l}$ as:

$$\boldsymbol{H}_{k,l} = \boldsymbol{B}_{k,l} \boldsymbol{A}_{k,l}^H \tag{1}$$

with a full rank matrices $\boldsymbol{B}_{k,l} \in \mathbb{C}^{N_{dl,k} \times r_{k,l}}$ and $\boldsymbol{A}_{k,l} \in \mathbb{C}^{N_{ul,l} \times r_{k,l}}$. We have $\boldsymbol{r}_{k,l}$ distinguishable significant paths contribute to $\boldsymbol{H}_{k,l}$, where distinguishable means with linearly independent antenna array responses from other paths, at both the Tx side and the Rx side.

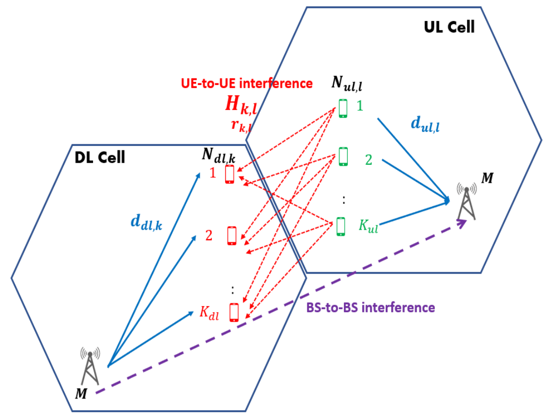

Fig. 1: DynTDD system model.

To analyze the UE-to-UE interference, we have both the DL and UL UEs will contribute to cancel each link of interference between them. We consider $\boldsymbol{F}_k \in \mathbb{C}^{N_{dl,k} \times d_{dl,k}}$ and $\boldsymbol{G}_l \in \mathbb{C}^{N_{ul,l} \times d_{ul,l}}$ as the Rx/Tx beamforming (BF) matrices at the kth DL and the lth UL users respectively. ZF from UL UE l to the DL UE k requires:

$$\boldsymbol{F}_k^H \boldsymbol{H}_{k,l} \boldsymbol{G}_l = 0, \forall k \in \{1, ..., K_{dl}\}, \forall l \in \{1, ..., K_{ul}\}\,. \tag{2}$$

Our system of Fig.1 is also called IBMAC (Interfering Broadcast–Multiple Access Channel) in [10] which corresponds to a two cell system with one cell being in DL (BC) and another in UL (MAC) and with interference between the two cells.

In our study, we suppose that the number of base station antennas is large enough so that all UL or DL UE streams can be supported, and that the BS to BS interference can be mitigated by exploiting a limited rank BS-to-BS channel [11]. Hence the IBMAC problem is then limited to the interference from UL users to DL users, which we may call IBMAC-IC (IBMAC Interference Channel). Regarding the number of data streams at Tx and Rx, we assume:

$$d_{dl,k} \geq 1 \quad \text{and} \quad d_{ul,l} \geq 1. \tag{3}$$

III. IA Feasibility Conditions for DynTDD UE-to-UE Reduced Rank MIMO IBMAC

In this section we analyze the overall UL UE to DL UE interference, considering deficient rank MIMO channels.

A. Proper Conditions

In [15] we have established the proper conditions, where the global proper conditions are given by [15, eq.(9)]. These necessary conditions themselves imply the following necessary condition [15, eq.(6)] (the proper condition is traditionally a single global condition requiring the number of variables to be greater than or equal to the number of constraints):

Theorem 1. ***Global Proper Condition for IA Feasibility in rank deficient MIMO IBMAC-IC***
For rank deficient MIMO channels, if the tuple of DoF $(d_{ul,1},...,d_{ul,K_{ul}},d_{dl,1},...,d_{ul,K_{dl}})$ is achievable through IA, then it must satisfy the global proper condition:

$$\begin{aligned}&\sum_{l=1}^{K_{ul}} d_{ul,l}(N_{ul,l}-d_{ul,l})+\sum_{k=1}^{K_{dl}} d_{dl,k}(N_{dl,k}-d_{dl,k})\\&\geq\sum_{l=1}^{K_{ul}}\sum_{k=1}^{K_{dl}}\min(r_{k,l}d_{dl,k},r_{k,l}d_{ul,l},d_{ul,l}d_{dl,k})\ .\end{aligned} \tag{4}$$

Note that this condition subsumes the SU MIMO conditions $d_{ul,l}\leq N_{ul,l}$, $d_{dl,k}\leq N_{dl,k}$ so that the number of variables on the LHS is non-negative. Apart from this proper condition for the overall system, we get an overall set of proper conditions by considering all subsystems also.

Theorem 2. ***Overall Proper Conditions for IA Feasibility in rank deficient MIMO IBMAC-IC***
The conditions in (4) *should be satisfied also by any subsystem, i.e. the IBMAC-IC formed by any subset of the UL users and any subset of the DL users.*

For a MIMO IBMAC-IC with full rank channels, the proper conditions in (4) hold with $r_{k,l}=\min\{N_{dl,k},N_{ul,l}\}\geq\max\{d_{dl,k},d_{ul,l}\}$. We have also considered sharing (distributing) IA between Rx and Tx for which the DoF are given by [15, eq.(26)] and [15, eq.(27)], or also one-sided ZF in [15, eq.(28)] and [15, eq.(29)].

B. Necessary and Sufficient Conditions for Regular Channels

In this sub-section we provide a detailed analysis of the interference by shedding light on the channel matrices and the Beamformers at Tx and Rx to find a solution for equation (2). These details for full rank channels will be useful for the understanding of the next sub-section where we shall consider reduced rank channel matrices. We revisit the feasibility analysis framework of [6], [5] and [4]. From the analysis in [6], [5], we know that the linear IA will be feasible for generic channels coefficients. An IA solution for channels $\boldsymbol{H}_{k,l}$ in (2) will be feasible if and only if a perturbed IA solution $\boldsymbol{F}_k+d\boldsymbol{F}_k$ and $\boldsymbol{G}_l+d\boldsymbol{G}_l$ exists for perturbed channels $\boldsymbol{H}_{kl}+d\boldsymbol{H}_{kl}$:

$$(\boldsymbol{F}_k^H+d\boldsymbol{F}_k^H)(\boldsymbol{H}_{k,l}+d\boldsymbol{H}_{k,l})(\boldsymbol{G}_l+d\boldsymbol{G}_l)=0\ . \tag{5}$$

The "if" part follows from considering that (2) becomes a special case of (5) when the perturbations disappear. The "only if" part follows from the philosophy of homotopy methods [16], [17] in which the solution of any instance of the problem (here for the given channel matrices) can be obtained by analytical continuation of the solution at any particular instance (which will correspond here to a particular choice of the channel matrices). The particular instance is typically chosen in a way to allow analytical problem solvability. The analytical continuation works as long as a Jacobian appearing in the problem continues to have full rank. This Jacobian will appear here also. For arbitrarily small perturbations (as in the homotopy method), expanding the products in (5) and considering only first order perturbations, we get:

$$\boldsymbol{F}_k^H\boldsymbol{H}_{k,l}\,d\boldsymbol{G}_l+d\boldsymbol{F}_k^H\boldsymbol{H}_{k,l}\boldsymbol{G}_l=-\boldsymbol{F}_k^H d\boldsymbol{H}_{k,l}\,\boldsymbol{G}_l \tag{6}$$

which need to be considered jointly for all interference links, and all Tx/Rx involved. (6) means that the feasibility of the bilinear equations (2) is equivalent to the feasibility of the linear equations (6), which can be rewritten jointly in the form $\boldsymbol{J}\boldsymbol{x}=-\boldsymbol{b}$. To identify the Jacobian $\boldsymbol{J}$, continue to consider link (k,l), for which we can obtain $\boldsymbol{J}_{kl}\boldsymbol{x}_{kl}=-\boldsymbol{b}_{kl}=-\text{vec}(\boldsymbol{F}_k^H d\boldsymbol{H}_{k,l}\,\boldsymbol{G}_l)$ by taking vec(.) of both sides of (6):

$$\text{vec}(\boldsymbol{F}_k^H\boldsymbol{H}_{k,l}\,d\boldsymbol{G}_l)=(\boldsymbol{I}_{d_{ul,l}}\otimes\boldsymbol{F}_k^H\boldsymbol{H}_{k,l})\,\text{vec}(d\boldsymbol{G}_l) \tag{7a}$$
$$\text{vec}(d\boldsymbol{F}_k^H\boldsymbol{H}_{k,l}\,\boldsymbol{G}_l)=((\boldsymbol{H}_{k,l}\boldsymbol{G}_l)^T\otimes\boldsymbol{I}_{d_{dl,k}})\,\text{vec}(d\boldsymbol{F}_k^H) \tag{7b}$$

Then we get for link (k,l) the system $\boldsymbol{J}_{kl}\boldsymbol{x}_{kl}=-\boldsymbol{b}_{kl}$ with

$$\boldsymbol{x}_{kl}=\begin{bmatrix}\text{vec}(d\boldsymbol{G}_l)^T & \text{vec}(d\boldsymbol{F}_k^H)^T\end{bmatrix}^T, \tag{8}$$

$$\boldsymbol{J}_{kl}=\begin{bmatrix}\boldsymbol{I}_{d_{ul,l}}\otimes\boldsymbol{F}_k^H\boldsymbol{H}_{k,l} & (\boldsymbol{H}_{k,l}\boldsymbol{G}_l)^T\otimes\boldsymbol{I}_{d_{dl,k}}\ .\end{bmatrix} \tag{9}$$

Now, the ZF conditions in (2) are insensitive to pre or post multiplication by non-singular square mixing matrices, or in other words, only the column spaces of the Rx/Tx filters $\boldsymbol{F}_k$, $\boldsymbol{G}_l$ matter. The actual available Rx/Tx variables are revealed by parameterizing the precoders and decoders as:

$$\boldsymbol{F}_k=\begin{bmatrix}\boldsymbol{I}_{d_{dl,k}}\\\overline{\boldsymbol{F}}_k\end{bmatrix},\boldsymbol{G}_l=\begin{bmatrix}\boldsymbol{I}_{d_{ul,l}}\\\overline{\boldsymbol{G}}_l\end{bmatrix} \tag{10}$$

where $\overline{\boldsymbol{F}}_k$ and $\overline{\boldsymbol{G}}_l$ are matrices of size $(N_{dl,k}-d_{dl,k})\times d_{dl,k}$ and $(N_{ul,l}-d_{ul,l})\times d_{ul,l}$ respectively, and which represent the only part of the $\boldsymbol{F}_k$, $\boldsymbol{G}_l$ that need/can be perturbed. They represent the variables appearing in the proper conditions. For channels with a continuous pdf, this parameterization is possible w.p. 1 and furthermore guarantees the Rx/Tx filters to have a rank equal to their number of streams d. Now, as in [6], [5], we can simplify the selected channels and associated ZF Rx/Tx filters around which we consider the perturbation. In particular we consider $\overline{\boldsymbol{F}}_k=\boldsymbol{0}$ and $\overline{\boldsymbol{G}}_l=\boldsymbol{0}$.

Now, the partitioning in $\boldsymbol{F}_k$, $\boldsymbol{G}_l$ leads to a corresponding channel partitioning:

$$\boldsymbol{H}_{kl} = \begin{bmatrix} \boldsymbol{H}_{kl}^{(1)} & \boldsymbol{H}_{kl}^{(2)} \\ \boldsymbol{H}_{kl}^{(3)} & \boldsymbol{H}_{kl}^{(4)} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{d_{dl,k} \times d_{ul,l}} & \boldsymbol{H}_{kl}^{(2)} \\ \boldsymbol{H}_{kl}^{(3)} & \mathbf{0}_{(N_{dl,k}-d_{dl,k}) \times (N_{ul,l}-d_{ul,l})} \end{bmatrix} \quad (11)$$

Indeed, with $\overline{\boldsymbol{F}}_k = \mathbf{0}$, $\overline{\boldsymbol{G}}_l = \mathbf{0}$, the ZF condition (2) becomes $\boldsymbol{F}_k^H \boldsymbol{H}_{k,l} \boldsymbol{G}_l = \boldsymbol{H}_{kl}^{(1)} = \mathbf{0}$. On the other hand, with these same Rx/Tx filters, the IA perturbation does not involve $\boldsymbol{H}_{kl}^{(4)}$ which we can hence take to be zero. Though the introduction of zeros in $\boldsymbol{H}_{kl}$ may lead to rank reduction, this has no effect as long as the resulting $\text{rank}(\boldsymbol{H}_{kl}) \geq \max(d_{dl,k}, d_{ul,l})$. Now we get for the perturbed system feasibility:

$$\begin{bmatrix} \boldsymbol{I}_{d_{ul,l}} \otimes \boldsymbol{H}_{kl}^{(2)} & \boldsymbol{H}_{kl}^{(3)T} \otimes \boldsymbol{I}_{d_{dl,k}} \end{bmatrix} \begin{bmatrix} \text{vec}(d\overline{\boldsymbol{G}}_l) \\ \text{vec}(d\overline{\boldsymbol{F}}_k^H) \end{bmatrix} = -\text{vec}(d\boldsymbol{H}_{kl}^{(1)}). \quad (12)$$

By considering all links, we get the overall system $\boldsymbol{J}\boldsymbol{x} = -\boldsymbol{b}$:

$$\boldsymbol{x}^T = \begin{bmatrix} \text{vec}^T(d\overline{\boldsymbol{G}}_1) \cdots \text{vec}^T(d\overline{\boldsymbol{G}}_{K_{ul}}) \text{vec}^T(d\overline{\boldsymbol{F}}_1^H) \cdots \text{vec}^T(d\overline{\boldsymbol{F}}_{K_{dl}}^H) \end{bmatrix}, \quad (13)$$

$$\boldsymbol{b}^T = \begin{bmatrix} \text{vec}^T(d\boldsymbol{H}_{11}) \cdots \text{vec}^T(d\boldsymbol{H}_{1,K_{ul}}) \cdots \text{vec}^T(d\boldsymbol{H}_{K_{dl},K_{ul}}) \end{bmatrix}, \quad (14)$$

and $\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_G \, \boldsymbol{J}_F \end{bmatrix} =$

$$\underbrace{\begin{bmatrix} \boldsymbol{I}_{d_{ul,1}} \otimes \boldsymbol{H}_{11}^{(2)} & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{0} & \boldsymbol{I}_{d_{ul,K_{ul}}} \otimes \boldsymbol{H}_{1K_{ul}}^{(2)} \\ \vdots & \vdots \\ \boldsymbol{I}_{d_{ul,1}} \otimes \boldsymbol{H}_{K_{dl}1}^{(2)} & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{0} & \boldsymbol{I}_{d_{ul,K_{ul}}} \otimes \boldsymbol{H}_{K_{dl}K_{ul}}^{(2)} \end{bmatrix}}_{\boldsymbol{J}_G} \underbrace{\begin{bmatrix} \boldsymbol{H}_{11}^{(3)\,T} \otimes \boldsymbol{I}_{d_{dl,1}} & \mathbf{0} \\ \vdots & \vdots \\ \boldsymbol{H}_{1K_{ul}}^{(3)\,T} \otimes \boldsymbol{I}_{d_{dl,1}} & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{0} & \boldsymbol{H}_{K_{dl}1}^{(3)\,T} \otimes \boldsymbol{I}_{d_{dl,K_{dl}}} \\ \vdots & \vdots \\ \mathbf{0} & \boldsymbol{H}_{K_{dl}K_{ul}}^{(3)\,T} \otimes \boldsymbol{I}_{d_{dl,K_{dl}}} \end{bmatrix}}_{\boldsymbol{J}_F} \quad (15)$$

The block $\boldsymbol{I}_{d_{ul,l}} \otimes \boldsymbol{H}_{kl}^{(2)}$ in $\boldsymbol{J}_G$ has dimensions $d_{ul,l}d_{dl,k} \times d_{ul,l}(N_{ul,l} - d_{ul,l})$, the block $\boldsymbol{H}_{kl}^{(3)\,T} \otimes \boldsymbol{I}_{d_{dl,k}}$ in $\boldsymbol{J}_F$ has dimensions $d_{ul,l}d_{dl,k} \times (N_{dl,k} - d_{dl,k})d_{dl,k}$.
The dimensions are for matrix $\boldsymbol{J}_G$:
$\sum_{l=1}^{K_{ul}} \sum_{k=1}^{K_{dl}} d_{ul,l}d_{dl,k} \times \sum_{l=1}^{K_{ul}} (N_{ul,l} - d_{ul,l})d_{ul,l}$,
for matrix $\boldsymbol{J}_F$:
$\sum_{l=1}^{K_{ul}} \sum_{k=1}^{K_{dl}} d_{ul,l}d_{dl,k} \times \sum_{k=1}^{K_{dl}} (N_{dl,k} - d_{dl,k})d_{dl,k}$.

Theorem 3. ***Necessary and Sufficient Condition for IA Feasibility in a Regular MIMO IBMAC-IC***
For a full rank MIMO IBMAC-IC, the DoF tuple $(d_{ul,1}, ..., d_{ul,K_{ul}}, d_{dl,1}, ..., d_{ul,K_{dl}})$ is feasible almost surely if and only if $\boldsymbol{J}$ has full row rank.

The proof appears in [10] and is also a special case of the rank deficient case considered below. Note that this full row rank requirement on $\boldsymbol{J}$ implies Theorem 2. [10] also finds sufficiency in the limited scenario in which all DL UEs and UL UEs have the same number of data streams $d_{dl,k} = d_{dl}, d_{ul,l} = d_{ul}$ and $N_{dl,k}$ and $N_{ul,l}$ must satisfy $mod(N_{dl,k} - d_{dl}, d_{ul}) = mod(N_{ul,l} - d_{ul}, d_{dl}) = 0$.

C. Necessary and Sufficient Conditions for Reduced Rank Channels

Existing work on IA feasibility assume only the full rank channel model [6], [5] and [4], but in many practical propagation environments such as the number of surrounding scatterers which is finite and limited, the MIMO channel matrix is likely to have reduced rank [18], [19], so thus designs based on full rank channels become inefficient. In this paper we consider the reduced rank channel model, which is a general case and the full rank channel is a special case of the reduced rank channel model. For the case of rank deficient interfering channels, we consider the channel factorization in (1), combined with the channel partitioning in (11), leading to:

$$\boldsymbol{A}_{kl}^H = \begin{bmatrix} \boldsymbol{A}_{kl}^{(1)} & \boldsymbol{A}_{kl}^{(2)} \end{bmatrix}, \; \boldsymbol{B}_{kl}^H = \begin{bmatrix} \boldsymbol{B}_{kl}^{(1)} & \boldsymbol{B}_{kl}^{(2)} \end{bmatrix}. \quad (16)$$

The matrix blocks $\boldsymbol{A}_{kl}^{(1)}$ and $\boldsymbol{B}_{kl}^{(1)}$ have dimensions $r_{kl} \times d_{ul,l}$ and $r_{kl} \times d_{dl,k}$ respectively. So (11) becomes:

$$\boldsymbol{H}_{kl} = \begin{bmatrix} \mathbf{0}_{d_{dl,k} \times d_{ul,l}} & \boldsymbol{B}_{kl}^{(1)H} \boldsymbol{A}_{kl}^{(2)} \\ \boldsymbol{B}_{kl}^{(2)H} \boldsymbol{A}_{kl}^{(1)} & \boldsymbol{B}_{kl}^{(2)H} \boldsymbol{A}_{kl}^{(2)} \end{bmatrix} \quad (17)$$

where again $\boldsymbol{H}_{kl}^{(4)} = \boldsymbol{B}_{kl}^{(2)H} \boldsymbol{A}_{kl}^{(2)}$ will not appear further in the analysis. Nevertheless, the structure in (1), (17) assumes that the following requirements are satisfied:

- To have $\boldsymbol{H}_{kl}^{(1)} = \boldsymbol{B}_{kl}^{(1)H} \boldsymbol{A}_{kl}^{(1)} = \mathbf{0}$, we can take $\boldsymbol{A}_{kl}^{(1)}$ with n_{kl} rows equal to zero, and $\boldsymbol{B}_{kl}^{(1)}$ with the complementary $r_{kl} - n_{kl}$ rows equal to zero.
- The channel model in (1) assumes $\text{rank}(\boldsymbol{A}_{kl}) = \text{rank}(\boldsymbol{B}_{kl}) = r_{kl}$.
- With $\text{rank}(\boldsymbol{A}_{kl}^{(1)}) = \min(r_{kl} - n_{kl}, d_{ul,l})$, $\boldsymbol{A}_{kl}^{(2)}$ should have the complementary rank to have $\text{rank}(\boldsymbol{A}_{kl}) = r_{kl}$. Hence the number of columns of $\boldsymbol{A}_{kl}^{(2)}$ needs to satisfy: $N_{ul,l} - d_{ul,l} \geq r_{kl} - \min(r_{kl} - n_{kl}, d_{ul,l})$,
- Same discussion for $\boldsymbol{B}_{kl}$, so we need to have $N_{dl,k} - d_{dl,k} \geq r_{kl} - \min(n_{kl}, d_{dl,k})$.

In what follows, we shall assume that all these conditions are met. On the other hand, in the rank deficient case, also the channel perturbation exhibits structure:

$$\text{vec}(d\boldsymbol{H}_{kl}^{(1)}) = \text{vec}(d\boldsymbol{B}_{kl}^{(1)H} \boldsymbol{A}_{kl}^{(1)}) + \text{vec}(\boldsymbol{B}_{kl}^{(1)H} \, d\boldsymbol{A}_{kl}^{(1)}) \quad (18)$$

Now exploiting the channel structure in (17), $\boldsymbol{J}_G$ and $\boldsymbol{J}_F$ in (15) can be written as:

$$\boldsymbol{J}_G = \begin{bmatrix} \boldsymbol{I}_{d_{ul,1}} \otimes \boldsymbol{B}_{11}^{(1)H} \boldsymbol{A}_{11}^{(2)} & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \boldsymbol{I}_{d_{ul,K_{ul}}} \otimes \boldsymbol{B}_{1K_{ul}}^{(1)H} \boldsymbol{A}_{1K_{ul}}^{(2)} \\ \vdots & & \vdots \\ \boldsymbol{I}_{d_{ul,1}} \otimes \boldsymbol{B}_{K_{dl}1}^{(1)H} \boldsymbol{A}_{K_{dl}1}^{(2)} & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \boldsymbol{I}_{d_{ul,K_{ul}}} \otimes \boldsymbol{B}_{K_{dl}K_{ul}}^{(1)H} \boldsymbol{A}_{K_{dl}K_{ul}}^{(2)} \end{bmatrix} \tag{19}$$

$$\boldsymbol{J}_F = \begin{bmatrix} (\boldsymbol{B}_{11}^{(2)H} \boldsymbol{A}_{11}^{(1)})^T \otimes \boldsymbol{I}_{d_{dl,1}} & & \boldsymbol{0} \\ \vdots & & \vdots \\ (\boldsymbol{B}_{1K_{ul}}^{(2)H} \boldsymbol{A}_{1K_{ul}}^{(1)})^T \otimes \boldsymbol{I}_{d_{dl,1}} & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & (\boldsymbol{B}_{K_{dl}1}^{(2)H} \boldsymbol{A}_{K_{dl}1}^{(1)})^T \otimes \boldsymbol{I}_{d_{dl,K_{dl}}} \\ \vdots & & \vdots \\ \boldsymbol{0} & & (\boldsymbol{B}_{K_{dl}K_{ul}}^{(2)H} \boldsymbol{A}_{K_{dl}K_{ul}}^{(1)})^T \otimes \boldsymbol{I}_{d_{dl,K_{dl}}} \end{bmatrix} \tag{20}$$

For $\boldsymbol{b}$ in $\boldsymbol{J}\,\boldsymbol{x} = -\boldsymbol{b}$, we consider the following vectorization:

$$vec(d\boldsymbol{H}_{kl}^{(1)}) = vec(\boldsymbol{B}_{kl}^{(1)H}\, d\boldsymbol{A}_{kl}^{(1)}) + vec(d\boldsymbol{B}_{kl}^{(1)H} \boldsymbol{A}_{kl}^{(1)}) = (\boldsymbol{I}_{d_{ul,l}} \otimes \boldsymbol{B}_{kl}^{(1)H})\, vec(d\boldsymbol{A}_{kl}^{(1)}) + (\boldsymbol{A}_{kl}^{(1)T} \otimes \boldsymbol{I}_{d_{dl,k}})\, vec(d\boldsymbol{B}_{kl}^{(1)H}) \tag{21}$$

Hence the vector $\boldsymbol{b}$ can be written as $\boldsymbol{b} = \boldsymbol{J}_H \boldsymbol{x}_H$ with:

$$\boldsymbol{J}_H = \underbrace{\begin{bmatrix} (\boldsymbol{I}_{d_{ul,1}} \otimes \boldsymbol{B}_{11}^{(1)H}) & \boldsymbol{0} & (\boldsymbol{A}_{11}^{(1)T} \otimes \boldsymbol{I}_{d_{dl,1}}) & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & (\boldsymbol{I}_{d_{ul,K_{ul}}} \otimes \boldsymbol{B}_{K_{dl}K_{ul}}^{(1)H}) & \boldsymbol{0} & (\boldsymbol{A}_{K_{dl}K_{ul}}^{(1)T} \otimes \boldsymbol{I}_{d_{dl,K_{dl}}}) \end{bmatrix}}_{\boldsymbol{J}_B \qquad\qquad \boldsymbol{J}_A} \tag{22}$$

$$\boldsymbol{x}_H^T = \left[vec(d\boldsymbol{A}_{11}^{(1)})^T \ldots vec(d\boldsymbol{A}_{K_{dl}K_{ul}}^{(1)})^T \; vec(d\boldsymbol{B}_{11}^{(1)})^T \; \ldots vec(d\boldsymbol{B}_{K_{dl}K_{ul}}^{(1)})^T \right] \tag{23}$$

For the purpose of further analysis, it may be of interest to note that we can write $\boldsymbol{J}$ as $\boldsymbol{J} = \boldsymbol{J}_H \boldsymbol{T}$ where $\boldsymbol{T}$ is given by :

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_A & 0 \\ 0 & \boldsymbol{T}_B \end{bmatrix} \tag{24}$$

$$\boldsymbol{T}_A = \begin{bmatrix} \boldsymbol{I}_{d_{ul,1}} \otimes \boldsymbol{A}_{11}^{(2)} & \boldsymbol{0} \\ \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{I}_{d_{ul,K_{ul}}} \otimes \boldsymbol{A}_{1K_{ul}}^{(2)} \\ \vdots & \vdots \\ \boldsymbol{I}_{d_{ul,1}} \otimes \boldsymbol{A}_{K_{dl}1}^{(2)} & \boldsymbol{0} \\ \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{I}_{d_{ul,K_{ul}}} \otimes \boldsymbol{A}_{K_{dl}K_{ul}}^{(2)} \end{bmatrix} \tag{25}$$

$$\boldsymbol{T}_B = \begin{bmatrix} \boldsymbol{B}_{11}^{(2)H} \otimes \boldsymbol{I}_{d_{dl,1}} & \boldsymbol{0} \\ \vdots & \vdots \\ \boldsymbol{B}_{1K_{ul}}^{(2)H} \otimes \boldsymbol{I}_{d_{dl,1}} & \boldsymbol{0} \\ \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{B}_{K_{dl}1}^{(2)H} \otimes \boldsymbol{I}_{d_{dl,K_{dl}}} \\ \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{B}_{K_{dl}K_{ul}}^{(2)H} \otimes \boldsymbol{I}_{d_{dl,K_{dl}}} \end{bmatrix} \tag{26}$$

Note the following dimensions:

- The blocks $(\boldsymbol{I}_{d_{ul,l}} \otimes \boldsymbol{B}_{kl}^{(1)H})$ in $\boldsymbol{J}_B$ has the dimension $d_{ul,l}d_{dl,k} \times d_{ul,l}r_{kl}$,
- The blocks $(\boldsymbol{A}_{kl}^{(1)T} \otimes \boldsymbol{I}_{d_{dl,k}})$ in $\boldsymbol{J}_A$ has the dimension $d_{ul,l}d_{dl,k} \times r_{kl}d_{dl,k}$,
- The blocks $\boldsymbol{I}_{ul,l} \otimes \; \boldsymbol{A}_{kl}^{(2)}$ in $\boldsymbol{T}_A$ has the dimension $d_{ul,l}r_{kl} \times d_{ul,l}(N_{ul,l} - d_{ul,l})$,
- The blocks $\boldsymbol{B}_{kl}^{(2)H} \otimes \boldsymbol{I}_{d_{dl,k}}$ in $\boldsymbol{T}_B$ has the dimension $r_{kl}d_{dl,k} \times (N_{dl,k} - d_{dl,k})d_{dl,k}$.

Now we define the augmented matrix $\boldsymbol{J}_J$ as:

$$\boldsymbol{J}_J = [\boldsymbol{J} \;\; \boldsymbol{J}_H] \tag{27}$$

Now we are ready to formulate the following result.

Theorem 4. ***Necessary and Sufficient Condition for IA Feasibility in Reduced Rank MIMO IBMAC-IC***
For a deficient rank MIMO IBMAC-IC, the DoF $(d_{ul,1}, ..., d_{ul,K_{ul}}, d_{dl,1}, ..., \; d_{ul,K_{dl}})$ *are feasible almost surely if and only if*

$$rank(\boldsymbol{J}) = rank(\boldsymbol{J}_J) = rank([\boldsymbol{J} \;\; \boldsymbol{J}_H]) \tag{28}$$

i.e., the column space of $\boldsymbol{J}_H$ *in* (22) *should be contained in the column space of* $\boldsymbol{J}$ *in* (15).

Note that is result is valid for *any* interference network, with appropriately defined matrices $\boldsymbol{J}$, $\boldsymbol{J}_H$.

Proof : We have the following condition from [20, page 12]: a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent (i.e., the system has at least one solution) if and only if rank($[\boldsymbol{A} \quad \boldsymbol{b}]$) = rank($\boldsymbol{A}$). So the existence of a solution for our system $\boldsymbol{Jx} = \boldsymbol{b}$ implies that rank($[\boldsymbol{J} \quad \boldsymbol{b}]$) = rank($\boldsymbol{J}$). For the rank deficient channel case, this becomes rank($[\boldsymbol{J} \quad \boldsymbol{J}_H \boldsymbol{x}_H]$) = rank($\boldsymbol{J}$) which should hold for any $\boldsymbol{x}_H$. Hence we require rank($[\boldsymbol{J} \quad \boldsymbol{J}_H]$) = rank($\boldsymbol{J}$).

IV. Results and Discussion

In Table I we evaluate the DoF of a uniform system ($N_{ul,l} = N_{ul}$, $N_{dl,k} = N_{dl}$, $d_{ul,l} = d_{ul}$, $d_{dl,k} = d_{dl}$, $r_{kl} = r$) with $N_{dl} = 4$, $N_{ul} = 6$, $K_{dl} = 4$ and $K_{ul} = 2$, for the different conditions established in [15] and the proper and sufficient conditions given by Theorem 2 and Theorem 4. In the following we give the description of each element in Table I, where a generic tuple $(d_{dl}, d_{ul}, d_{tot})$ denotes the uniform DoF of a DL UE, an UL UE, and the overall UL+DL sum DoF:

- $(d_{p,dl}, d_{p,ul}, d_{p,tot})$ considering Theorem 2 in the centralized case, i.e. considering (only) the proper (necessary) IA feasibility conditions for a centralized design,
- $(d_{d,dl}, d_{d,ul}, d_{d,tot})$ considering the distributed method, with DL UE DoF as in [15, eq. (31a)], UL UE DoF as in [15, eq. (31b)] (with the corresponding optimized n_F, n_G shown in Table I and denoted as n_{F_d}, n_{G_d}), i.e. this is the distributed method in which Tx/Rx filters only depend on the low rank channel factors on their side (and are independent of the filter values on the other side, their design is closed-form, non-iterative), with an optimization of the distribution of the ZF roles among Tx/Rx,
- $(d_{c,dl}, d_{c,ul}, d_{c,tot})$ considering the combined method, with DL UE DoF as in [15, eq. (26)], the UL UE as in [15, eq. (27)] (with the corresponding optimized n_F, n_G shown in Table I and denoted as n_{F_c}, n_{G_c}), i.e. this concerns a feasible centralized approach in which there is an optimized partitioning of the ZF roles among all Tx/Rx: each interference link is either ZF'd by the Tx or the Rx involved (but the resulting Tx depend on the Rx and vice versa, the Tx/Rx design requires an iterative algorithm),
- $(d_{r,dl}, d_{r,ul}, d_{r,tot})$ considering Rx side ZF only as in [15, eq. (26)] with $n_F = K_{ul}$, i.e. all ZF is done by the Rx only (closed-form solutions, non-iterative, hence can be considered a distributed approach),
- $(d_{t,dl}, d_{t,ul}, d_{t,tot})$ considering Tx side ZF only as in [15, eq. (27)] with $n_G = K_{dl}$, i.e. all ZF is done by the Tx only (closed-form solutions, non-iterative, hence can be considered a distributed approach),
- $(d_{T4,dl}, d_{T4,ul}, d_{T4,tot})$ considering Theorem 4, i.e. the exactly maximally feasible DoF in a centralized approach (requires an iterative Tx/Rx design).

For the application of Theorem 4, we perform an algorithm that allows us to check the rank of the matrices $\boldsymbol{J}$ and $\boldsymbol{J}_J$ depending on the variables N_{ul}, N_{dl}, d_{ul}, d_{dl} and r_{kl}, when given the IC matrix $\boldsymbol{H}_{k,l}$ with random coefficients that must satisfy the conditions mentioned in subsection III-C. We test all possible combinations regarding the values of n_{kl} and also the possible positions of the zero rows in $\boldsymbol{A}_{kl}^{(1)}$ and $\boldsymbol{B}_{kl}^{(1)}$.

r	0	1	2	3	4
$(d_{p,dl}, d_{p,ul}, d_{p,tot})$	(4,6,28)	(3,4,20)	(2,4,16)	(2,2,12)	(2,2,12)
$(d_{d,dl}, d_{d,ul}, d_{d,tot})$	(4,6,28)	(3,4,20)	(2,2,12)	(0,3,6)*	(0,2,4)*
$(n_{F,d}, n_{G,d})$	(1,2)	(1,2)	(1,2)	(2,1)	(2,1)
$(d_{c,dl}, d_{c,ul}, d_{c,tot})$	(4,6,28)	(3,4,20)	(2,2,12)	(2,2,12)	(2,2,12)
$(n_{F,c}, n_{G,c})$	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
$(d_{r,dl}, d_{r,ul}, d_{r,tot})$	(4,6,28)	(2,6,20)	(2,1,10) (0,6,12)*	(2,1,10) (0,6,12)*	(2,1,10) (0,6,12)*
$(d_{t,dl}, d_{t,ul}, d_{t,tot})$	(4,6,28)	(4,2,20)	(1,2,8) (4,0,16)*	(1,2,8) (4,0,16)*	(1,2,8) (4,0,16)*
$(d_{T4,dl}, d_{T4,ul}, d_{T4,tot})$	(4,6,28)	(3,4,20)	(2,3,14)	(2,2,12)	(2,2,12)

TABLE I: DoF per user as a function of the rank of any cross link channel with $N_{ul} = 6$, $N_{dl} = 4$, $K_{ul} = 2$ and $K_{dl} = 4$.

- (*): the given DoF does not satisfy the conditions in (3), if a negatif DoF results from a formula, this DoF will be set to zero logically.

From the Table I and all the conditions previously established for IA Feasibility in [15], we observe the following points:

- The first line of Table I gives the proper condition, which represents an upper bound for the DoF, that is not necessarily reachable regarding the IA feasibility,
- The DoF given by the distributed design in the second line of Table I, is a feasible DoF for IA in such a model. The distributed solution does not exhibit sub-optimality w.r.t. the centralized optimal solution for $r \leq d$,
- The combined solution is given in the fourth line of Table I, both UL and DL UEs contribute to the IA in centralized design. This solution can reach the proper condition for $r > d$,
- The unilateral solution in the sixth and seventh line of Table I can penalize the UEs in charge of the ZF with a zero DoF, when attempting DoF maximization,
- From the results given in this table, we notice that, beyond a certain value of the IC rank, the DoF given by our sufficient condition (Theorem 4) provides a DoF, sometimes greater, but never smaller, than the other zero-forcing methods, even the distributed method (for which the DoF are feasible).

V. Conclusions

In this paper we address the rank deficient MIMO IBMAC Interference Channel, and we analyze the necessary and sufficient conditions for IA feasibility captured by the rank of a Jacobian matrix. A new theorem (Theorem 4) on the rank deficient MIMO IC DoF feasibility is provided, which is applicable to *any* MIMO interference network.

The extension of the full rank MIMO case, Theorem 3, to the rank deficient MIMO case, Theorem 4, is not as trivial as it may appear, requiring an original judicious simplified parameterization of rank deficient channels. We evaluate the DoF for a number of dimensions and compare the results between the proper conditions mentioned before in our previous work [15] including the centralized and two distributed designs, the zero-forcing in a Tx/Rx shared fashion or only one-sided considering Tx or Rx, and our necessary and sufficient condition given by Theorem 4. The numerical evaluations show that Theorem 4, which corresponds to a centralized design, gives greater DoF values than the ones given by the distributed designs, for which the conditions are also necessary and sufficient. In our future work, we will be addressing tighter necessary conditions than the proper conditions, and general sufficient conditions regarding the number of antennas, the rank of the channel matrix and the DoF, which may be loose compared to Theorem 4, but which will simplify the feasible DoF finding as they will not need the rank test of Theorem 4.

Acknowledgements

EURECOM's research is partially supported by its industrial members: ORANGE, BMW, SAP, iABG, Norton LifeLock, and by the projects EEMW4FIX, CellFree6G (French ANR).

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