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Throughput bound minimization for random access channel assignment

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***Abstract*—Throughput extremization is an important facet of performance modeling for low-power wide-area network (LPWAN) wireless networks (e.g., LoRaWAN) as it provides insight into the best and worst case behavior of the network. Our previous work on throughput extremization established lower and upper bounds on throughput for random access channel assignment over a collision erasure channel in which the lower bound is expressed in terms of the number of radios and sum load on each channel. In this paper the lower bound is further characterized by identifying two local minimizers (a load balanced assignment and an imbalanced assignment) where the decision variables are the number of radios assigned to each channel and the total load on each channel. A primary focus is to characterize how macro-parameters of the optimization, i.e., the total number of radios, their total load, and the minimum load per radio, determine the regions under which each of the local minimizers is in fact the global minimizer.**

***Index Terms*—wireless; random access; throughput; channel assignment; optimization.**

I. Introduction

Hallmark traits of wide-area Internet of Things (IoT) networks include: $i)$ a large number of low cost, low power, and long range radios, utilizing random access transmissions to an access point (AP) and $ii)$ the opportunity for the AP to assign these radios to separate wireless channels. As the offered load increases, it is evident that the system performance, measured in this paper as the AP's expected throughput, will depend upon how the radios are assigned to the available channels.

The investigation is motivated in part by currently deployed IoT systems commonly termed Low Power Wide Area Networks (LPWANs). The LoRaWAN random access medium access control (MAC) protocol, for example, supports 64 channels, each with a spreading factor between 7 and 12.

A. System model and results summary

Extending prior work by the authors [1], [2], the following system aspects are assumed: (see Fig. 1):

- The AP assigns each radio to one channel, and the channels are independent and identically distributed;
- Each radio has an infinite backlog of packets for uplink to the AP, i.e., the transmission queue is never empty;

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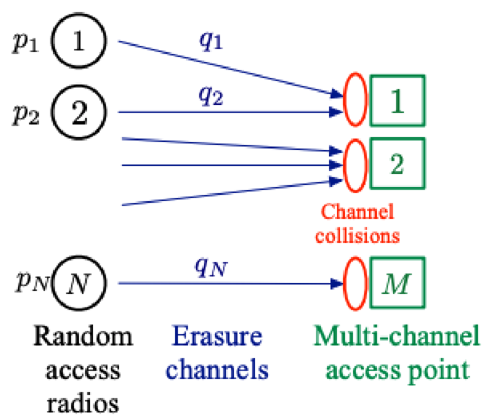

Fig. 1. The multi-channel random access erasure collision channel consists of $i)$ N radios employing random access transmissions, with contention probability p_i; $ii)$ each radio assigned to one of M uplink erasure channels with nonerasure probability q_{ij}; and $iii)$ a single AP at which simultaneous arrivals of multiple packets on a channel are lost, but single packet arrivals on separate channels are successfully received.

- The contention and message nonerasure probability of each radio on the network is independent across radios, channels, and time, and independent across time;
- All transmission attempts are synchronized to a common clock and have common duration;
- The arrival at the AP of multiple packets on a given channel leads to collision and loss of all such packets, but single packet arrivals on separate channels are each successfully received.

Motivated by providing more tractable lower and upper bounds on the difficult combinatorial optimization problem of assigning heterogeneous users (radios) to channels so as to extremize throughput of the given multi-channel random access system (§III-C), the paper is focused on the minimization of the average per-channel expected throughput lower bound $\underline{T}(n, \mu)$ in Cor. 1. Specific contributions include:

- Partial characterization of the optimal quasi-uniform allocation in the many small users regime (Prop. 2);
- Partial characterization of optimality between balanced and imbalanced allocations over two channels (Prop. 5).

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B. Previous and related work

Previous work by the authors focused on i) extremizing throughput bounds for a single channel $M = 1$ [1] and ii) finding preliminary results for the two channel scenario [2]. The latter paper introduced both lower and upper bounds on the system throughput which translated the associated channel assignment problem from a combinatorial problem to a lower dimensional nonlinear program (NLP). The contribution of this paper is to partially characterize the minimization of the lower bound on system throughput.

Related work on throughput optimization of multi-channel random access systems is too extensive to meaningfully summarize here, and so only a small number of the most relevant references known to the authors are mentioned here. Maximizing throughout through distributed price signalling in non-cooperative game was investigated in [3]. Slotted Aloha in which a radio continues to transmit until a collision occurs is considered in [4]. A seminal work on coordinated and uncoordinated throughput maximization, focused on machine to machine (M2M) communications is presented in [5]. Furthermore, a joint optimization technique based on spreading factor (SF) assignment, energy harvesting (EH) time duration and transmit power to maximize minimum throughput in LoRa networks was investigated in [6]. A Bayesian online backoff algorithm was proposed for studying throughput and random access delay distribution of unslotted ALOHA systems in LPWAN setting in [7]. An optimal resource allocation policy based on spectrum map and radio conditions in the LoRaWAN setting is studied in [8]. A mathematical model which accurately estimates how packet error rate depends on the offered load is proposed in [9]. Network level performance under the ALOHA protocol using autonomous sensing scheme allowing independent transmissions of CR users is investigated in [10]. Random Access Technology (RAT) selection for heterogeneous networks (HetNet) using a non-cooperative game framework is proposed in [11].

C. Outline

The rest of this paper is organized as follows: the system model is defined in §II, the throughput bounds are reviewed in §III, throughput minimization in the many small users regime is addressed in §IV, minimization over $M = 2$ channels is addressed in §V, numerical results are given in §VI, and a conclusion is in §VII. Some proofs are in the Appendix. Table I summarizes the most common notation.

TABLE I
SUMMARY OF NOTATION.

Notation	Interpretation
M	# channels provided by access point (AP)
N	# users to be served by AP
Σ	sum offered load across all users
$R \equiv \Sigma/M$	average per-channel load
$\underline{X}$	minimum load of a user
$\bar{X}$	maximum load of a user
$\mathcal{N}_j$	users i assigned to channel j
$x_j = (x_{ij}, i \in \mathcal{N}_j)$	loads of users assigned to channel j
$\tau(x_j)$	expected throughput on channel j
$\mu(x_j)$	average per-user load on channel j
$\pi(x_j)$	congestion on channel j
$\underline{\tau}, \bar{\tau}$	bounds on per-channel throughput
$x = (x_j, j \in [M])$	user loads assigned to each channel
$T(x)$	average per-channel expected throughput
$\underline{T}, \bar{T}$	bounds on average per-channel throughput
(n, μ)	# users and average load for each channel
$\mathcal{A}$	feasible set of (n, μ) chanel assignments
$\mathsf{P}_{\min}, \mathsf{P}_{\max}$	combinatorial channel assignment problems
$\underline{\mathsf{P}}, \bar{\mathsf{P}}$	nonlinear optimization problems

II. The random access erasure collision channel

The model is from [1], [2]. RV denotes random variable and IID denotes independent and identically distributed.

General Notation. Let $a \equiv b$ denote equal by definition. Write $[m:n] \equiv \{m, .., n\}$ for $m, n \in \mathbb{N}$ and $[n]$ for $[1:n]$.

Radios, users, and uplink channels. There are $N \in \mathbb{N}$ radios, indexed by $i \in [N]$, each with a wireless uplink to a shared access point (AP) or base station (BS). Radios are henceforth termed *users*. The users and AP have $M \in \mathbb{N}$ independent and identically distributed uplink channels. The AP assigns a channel $j \in [M]$ to each user $i \in [N]$ and (user, channel) indices are denoted as (i, j). All packets are the same size, all transmission have the same duration, and time is slotted into synchronized packet transmission slots.

Channel erasures. The uncertainty of the wireless channel is modeled by Bernoulli RVs where $q_i \in (0, 1)$ is the probability of non-erasure of a transmission from user i on any channel j, i.e., any (i, j) message arrives intact at the AP with probability q_i or is corrupted / dropped / lost with probability $1 - q_i$. Erasure channels are independent across users, channels, and time, and are identically distributed across channels and time.

Channel collisions. The AP is subject to collisions in each time slot on each channel, i.e., i) multiple packets arriving in the same time slot on the same channel are all lost, but ii) all packets that are the unique packet arriving on their channel in a time slot are successfully received.

Contention. Each user $i \in [N]$ is assumed to have a infinite-backlog of packets for transmission and random access channel contention. The decision to access the channel randomly by a user is IID across time and decisions in a given time slot are independent. The contention probability for user i is defined as $p_i \in (0, 1)$ and $p = (p_i, i \in [N])$ is the contention probability vector for N users. The contention probability is a design parameter to be optimized in order to extremize the throughput. As shown in §III-A, the parameter p_i is wrapped into design parameter x_i.

III. Throughput and throughput bounds

The throughput bounds below are from prior work [1], [2].

A. Single channel throughput

Let $\mathcal{N}_j \equiv (i_1, \ldots, i_{n_j}) \subset [N]$ be the indices of the n_j users assigned to channel $j \in [M]$. For user $i \in \mathcal{N}_j$, the probability of a message successfully arriving at the AP is $r_{ij} \equiv p_{ij} q_{ij}$. The expected throughput of the AP on channel

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j is the probability of a single message arriving at the AP on that channel in a given time slot, i.e.,

$$\tau(r_j) \equiv \sum_{i \in \mathcal{N}_j} r_{ij} \prod_{i' \in \mathcal{N}_j \setminus i} (1 - r_{i'j}) \tag{1}$$

with $r_j \equiv (r_{ij}, i \in \mathcal{N}_j)$. The change of variables

$$x_{ij} \equiv \frac{r_{ij}}{1 - r_{ij}}, \; i \in \mathcal{N}_j, \tag{2}$$

describes the offered load x_{ij} from user i on channel j and $x_j \equiv (x_{ij}, i \in \mathcal{N}_j)$ is the collection of user loads on the channel. The channel j expected throughput in terms of x_j is

$$\tau(x_j) \equiv \frac{n_j \mu(x_j)}{\pi(x_j)} = \frac{n_j \mu_j}{\pi_j} \tag{3}$$

where

$$\mu_j = \mu_j(x_j) \equiv \frac{1}{n_j} \sum_{i \in \mathcal{N}_j} x_{ij}, \quad \pi_j = \pi(x_j) \equiv \prod_{i \in \mathcal{N}_j} (1 + x_{ij}). \tag{4}$$

The quantities μ_j, π_j are the *average per-user (offered) load* and *channel congestion* on channel j, respectively. Furthermore, define the minimum $\underline{x}_j \equiv \min_i x_{ij}$ and maximum $\bar{x}_j \equiv \max_i x_{ij}$ individual user load on each channel j.

Proposition 1 ([1])**.** *With the notation above, the single channel throughput has lower and upper bounds*

$$\underline{\tau}(n_j, \mu_j) \leq \tau_j(x_j) \leq \bar{\tau}(n_j, \mu_j, \underline{x}_j, \bar{x}_j), \tag{5}$$

where,

$$\begin{aligned} \underline{\tau}(n_j, \mu_j) &\equiv \frac{n_j \mu_j}{(1 + \mu_j)^{n_j}} \qquad (6) \\ \bar{\tau}(n_j, \mu_j, \underline{x}_j, \bar{x}_j) &= \frac{n_j \mu_j}{(1 + \underline{x}_j)^{n_j \frac{\bar{x}_j - \mu_j}{\bar{x}_j - \underline{x}_j}} + (1 + \bar{x}_j)^{n_j \frac{\mu_j - \underline{x}_j}{\bar{x}_j - \underline{x}_j}}} \end{aligned}$$

Remark 1 (Bound interpretation)**.** *The lower bound asserts throughput is minimized by load homogenization: given only summary statistics (n_j, μ_j) on the actual heterogeneous per-user loads x_j, the worst-case throughput is achieved by the homogenized load in which each user has identical load μ_j. Similarly, the upper bound asserts throughput is maximized by load extremization: given only summary statistics $(n_j, \mu_j, \underline{x}_j, \bar{x}_j)$ on the actual heterogeneous per-user loads x_j, the best-case throughput is achieved by setting all users to have load either $\underline{x}_j$ or $\bar{x}_j$, with the number of each type such that there are n_j users with average per-user load μ_j.*

B. Average multiple channel throughput

Recall the per-channel quantities from §III-A: $(n_j, x_j, \mu_j, \underline{x}_j, \bar{x}_j)$. Collect them into lists, for $j \in [M]$:

$$n \equiv (n_j), \; x \equiv (x_j), \; \mu \equiv (\mu_j), \; \underline{x} \equiv (\underline{x}_j), \; \bar{x} \equiv (\bar{x}_j). \tag{7}$$

As channels are homogeneous and independent, it follows that the expected throughput per channel of the AP is:

$$T(x) \equiv \frac{1}{M} \sum_{j \in [M]} \tau(x_j). \tag{8}$$

The following corollary is immediate.

Corollary 1 ([2])**.** *With the notation above, the AP's average per-channel expected throughput has lower and upper bounds*

$$\underline{T}(n, \mu) \leq T(x) \leq \bar{T}(n, \mu, \underline{x}, \bar{x}). \tag{9}$$

where

$$\begin{aligned} \underline{T}(n, \mu) &\equiv \frac{1}{M} \sum_{j \in [M]} \underline{\tau}(n_j, \mu_j) \\ \bar{T}(n, \mu, \underline{x}, \bar{x}) &\equiv \frac{1}{M} \sum_{j \in [M]} \bar{\tau}(n_j, \mu_j, \underline{x}_j, \bar{x}_j). \end{aligned} \tag{10}$$

Observe, by construction, $N = \sum_{j \in [M]} n_j$ for any channel assignment $\mathcal{N}$. Define the sum, minimum, and maximum offered load $(\Sigma(x), \underline{X}(x), \bar{X}(x))$, where

$$\begin{aligned} \Sigma = \Sigma(x) &\equiv \sum_{j \in [M]} n_j \mu_j = \sum_{ij} x_{ij} \\ \underline{X} = \underline{X}(x) &\equiv \min_{j \in [M]} \underline{x}_j = \min_{ij} x_{ij} \\ \bar{X} = \bar{X}(x) &\equiv \max_{j \in [M]} \bar{x}_j = \max_{ij} x_{ij} \end{aligned} \tag{11}$$

System parameters $(M, N, \Sigma, \underline{X}, \bar{X})$ satisfy

$$1 \leq M < N, \quad 0 \leq \underline{X} \leq \frac{\Sigma}{N} \leq \bar{X}. \tag{12}$$

In words: $i)$ $\Sigma / N \in [\underline{X}, \bar{X}]$ ensures Σ is sufficient to serve all N users a load of at least $\underline{X}$ but no more than $\bar{X}$, and $ii)$ $M < N$ is because it is trivial to avoid channel contention when $N \leq M$ by assigning each user to its own channel.

C. Assignment and bound extremization

It is of interest to understand the impact of channel assignment on $T(x)$ (8). As described, the AP has M channels available to serve N users, where $N > M$ and typically $N \gg M$.

The N users have offered loads $x = (x_i, i \in [N])$, and an assignment $\mathcal{N} = (\mathcal{N}_j, j \in [M])$ partitions $[N]$ into M disjoint subsets, resulting in per-channel offered loads $x^{\mathcal{N}} = (x_j^{\mathcal{N}}, j \in [M])$, which in turn results in per-channel expected throughputs $(\tau(x_j^{\mathcal{N}}), j \in [M])$ in (3) and a sum expected throughput $T(x^{\mathcal{N}})$ in (8). As the assignment $\mathcal{N}$ determines the average per-channel expected throughput $T(x^{\mathcal{N}})$, the channel assignment problems are the combinatorial optimizations

$$\mathsf{P}_{\min} : \min_{\mathcal{N}} T(x^{\mathcal{N}}) \;\; \mathsf{P}_{\max} : \max_{\mathcal{N}} T(x^{\mathcal{N}}). \tag{13}$$

The simple upper bound on the number of possible assignments is N^M, which is infeasibly large even for moderate values of (M, N) in practical scenarios. Structural properties of the expected sum throughput function $T(\cdot)$ that would enable an efficient solution or approximation of problems $\mathsf{P}_{\min}, \mathsf{P}_{\max}$ may exist but are not evident to the authors. Instead, the authors analyze the nonlinear optimization problems associated with the expected sum throughput bounds. The system parameters $(M, N, \Sigma, \underline{X}, \bar{X})$, viewed as given

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exogenously, determine the set of feasible (n, μ) values, as defined below.

Definition 1. *Given system parameters $(M, N, \Sigma, \underline{X}, \bar{X})$ obeying (12), the set $\mathcal{A}$ holds all feasible (n, μ) values:*

$$\mathcal{A} = \Bigg\{ (n,\mu) : \sum_{j\in[M]} n_j = N, \ \sum_{j\in[M]} n_j\mu_j = \Sigma, \quad (14)$$
$$\underline{X} \leq \min_{j\in[M]} \mu_j \leq \max_{j\in[M]} \mu_j \leq \bar{X}, \ 1 \leq \min_{j\in[M]} n_j \Bigg\}.$$

The requirement $1 \leq \min_{j\in[M]} n_j$ ensures each channel is utilized by at least one user. With the feasible set $\mathcal{A}$ defined, it is natural to consider the extremization of the bounds on the average per-channel expected throughput:

$$\underline{\mathsf{P}} : \min_{n,\mu\in\mathcal{A}} \underline{T}(n,\mu) \quad \bar{\mathsf{P}} : \max_{n,\mu\in\mathcal{A}} \bar{T}(n,\mu). \tag{15}$$

The four problems $(\underline{\mathsf{P}}, \mathsf{P}_{\min}, \mathsf{P}_{\max}, \bar{\mathsf{P}})$ have ordered values:

$$\min_{n,\mu\in\mathcal{A}} \underline{T}(n,\mu) \leq \min_{\mathcal{N}} T(x^{\mathcal{N}}) \leq \max_{\mathcal{N}} T(x^{\mathcal{N}}) \leq \max_{n,\mu\in\mathcal{A}} \bar{T}(n,\mu). \tag{16}$$

This paper offers partial analysis of the minimization of $\underline{\mathsf{P}}$ under different regimes of the exogenous parameters $(N, M, \bar{X})$ in the following sections.

IV. Throughput lower bound minimization in the many small users regime

The many small users regime is defined as the limit as $N \uparrow \infty$ while holding (M, Σ) fixed. The term small is due to the average load per user Σ/N vanishing to 0. This section is organized as follows: a parameterized set of feasible points is defined in §IV-A, the corresponding parameterized asymptotic throughput is given in §IV-B, and a partial characterization of the optimal point within the class is given in §IV-C.

A. A class of feasible points parameterized by K

Fix system parameters $(\Sigma, M, \underline{X})$, obeying (12). The following class of feasible points, puts the minimum number of users (one) and the minimum per-user load ($\underline{X}$) on K of the M channels, and balances the remaining $N - K$ users and $\Sigma - K\underline{X}$ load across the remaining $M - K$ channels.

Definition 2 (Quasi-uniform feasible points)**.** *For $K \in [0 : M-1]$, the quasi-uniform allocation (n^K, μ^K) is*

$$(n_j^K, \mu_j^K) = \begin{cases} (1, \underline{X}), & j \in [K] \\ \left(\frac{N-K}{M-K}, \frac{\Sigma-K\underline{X}}{N-K}\right), & j \in [K+1 : M] \end{cases} \tag{17}$$

Such points are always feasible with respect to Def. 1:

$$\sum_{j\in[M]} n_j^K = K \cdot 1 + (M-K) \cdot \frac{N-K}{M-K} = N \tag{18}$$
$$\sum_{j\in[M]} n_j^K \mu_j^K = K 1 \underline{X} + (M-K)\frac{N-K}{M-K}\frac{\Sigma - K\underline{X}}{N-K} = \Sigma$$

Remark 2. *The focus on quasi-uniform points is motivated in part by results in §V where, for $M = 2$ channels, the throughput lower bound $\underline{T}$ is stationary for allocations that are either load balanced (i.e., $K = 0$) or imbalanced (i.e., $K = M-1$). Future work will seek to extremize $\underline{T}$ over a broader class of points.*

B. Throughput in the asymptotic many small users regime

Per (10) and Def. 2, $\underline{T}$ at (n^K, μ^K), denoted $\underline{T}(K)$, is

$$\underline{T}(K) \equiv \underline{T}(n^K, \mu^K) = \frac{1}{M}\left[\frac{\underline{X}K}{1+\underline{X}} + \frac{\Sigma - \underline{X}K}{\left(1 + \frac{\Sigma - \underline{X}K}{N-K}\right)^{\frac{N-K}{M-K}}}\right]. \tag{19}$$

Using $\lim_{n\uparrow\infty}(1 + 1/n)^n = \mathrm{e}$ yields the many small users asymptotic average per-channel throughput, denoted $\underline{T}^{\uparrow}(K)$:

$$\begin{aligned} \underline{T}^{\uparrow}(K) &\equiv \lim_{N\uparrow\infty} \underline{T}^K = \frac{1}{M}\left[\frac{K\underline{X}}{1+\underline{X}} + (\Sigma - \underline{X}K)\mathrm{e}^{-\frac{\Sigma-\underline{X}K}{M-K}}\right] \\ &= \frac{M-K}{M}\left[\frac{\underline{X}}{1+\underline{X}}\frac{K}{M-K} + \frac{\Sigma - \underline{X}K}{M-K}\mathrm{e}^{-\frac{\Sigma-\underline{X}K}{M-K}}\right] \end{aligned} \tag{20}$$

Remark 3 (Feasibility and approximation accuracy)**.** *With $(M, \Sigma, \underline{X}, K)$ fixed, it is evident that $i)$ the quasi-uniform allocations in Def. 2 are feasible for $N \leq \Sigma/\underline{X}$ but infeasible for $N > \Sigma/\underline{X}$, as the per user load in channels $K+1$ through M falls below the minimum of $\underline{X}$, and $ii)$ the accuracy of the approximation $\underline{T}(K) \approx \underline{T}^{\uparrow}(K)$ is increasing in N.*

The highest accuracy is at the largest feasible value of N, i.e., $N = \Sigma/\underline{X}$, where the accuracy of the throughput approximation in channels $K+1$ through M is tied to the accuracy of the approximation $(1+\underline{X})^{1/\underline{X}} \approx \mathrm{e}$. The accuracy of the latter is decreasing in $\underline{X}$ over $(0, 1]$, from perfect accuracy as $\underline{X} \downarrow 0$ to a ratio of $2/\mathrm{e} \approx 0.73$ at $\underline{X} = 1$.

The lowest accuracy is at the smallest feasible value of N, i.e., $N = M$, where the accuracy of the throughput approximation in channels $K+1$ through M is tied to the accuracy of the approximation $(1+x)^{1/x} \approx \mathrm{e}$ at $x = x(K) = \frac{\Sigma - K\underline{X}}{M-K}$. The accuracy of the latter is decreasing in x, and, as $x(K)$ is increasing in K, the accuracy is highest for K small.

In summary, the approximation accuracy is highest for N large (i.e., near $\Sigma/\underline{X}$) and improves when $\underline{X}$ is small, and lowest for N small (i.e., near M), but improves for K small.

Remark 4 (Connection with classic slotted Aloha throughput analysis)**.** *The value $\underline{T}^{\uparrow}(0)$ corresponds to load balancing across all M channels; the resulting asymptotic average per-channel throughput is the classic throughput of slotted Aloha*

$$\underline{T}^{\uparrow}(0) = R\mathrm{e}^{-R}, \quad R \equiv \Sigma/M. \tag{21}$$

Viewing the asymptotic throughput as a function of the sum offered load per channel $R \equiv \Sigma/M$, an elementary analysis yields the conclusion that the asymptotic average per-channel throughput is initially increasing then subsequently decreasing in R, reaching the maximum asymptotic throughput of $1/\mathrm{e}$ at $R = 1$, i.e., when the offered load per channel is unity.

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C. Partial characterization of extremal quasi-uniform points

Recall $R \equiv \Sigma/M$ is the average per-channel load. Although $K \in [0 : M-1]$ is integer-valued, for purpose of optimization it will be treated as real-valued over the interval $[0, M-1]$. A change of variables from K to Y is introduced:

$$Y = \frac{\Sigma - \underline{X}K}{M-K}, \text{ i.e., } K = \frac{MY - \Sigma}{Y - \underline{X}}. \quad (22)$$

Per Def. 2, $Y(K)$ is the average per-channel load over the $M-K$ channels not assigned the minimum per-channel load of $\underline{X}$. Observe Y has range $[R, \Sigma - \underline{X}(M-1)]$ and, as $\Sigma > M\underline{X}$ (12), $\underline{X} < R$ is not in this range.

Substitution of the variable change from K to Y (22) into $\underline{T}^{\uparrow}(K)$ (20) yields, with $R \equiv \Sigma/M$

$$\underline{T}^{\uparrow}(Y) = \frac{R-\underline{X}}{Y-\underline{X}} \left[\left(\frac{\underline{X}}{1+\underline{X}} \right) \left(\frac{Y-R}{R-\underline{X}} \right) + Y \mathrm{e}^{-Y} \right] \quad (23)$$

Note $\underline{T}^{\uparrow}(Y)$ has a pole at $Y = \underline{X}$ but, per above, $\underline{X}$ is not in the range of Y, so $\underline{T}^{\uparrow}(Y)$ is finite-valued for all feasible Y.

The following result gives a partial characterization of the value of K that extremizes $\underline{T}^{\uparrow}(K)$.

Proposition 2. *The function $\underline{T}^{\uparrow}(Y)$ over $Y \in [R, \Sigma - \underline{X}(M-1)]$ may be either increasing in Y, decreasing in Y, or may have one or more stationary points. These stationary points, if any, are the set of intersections*

$$g_q(Y; \underline{X}) = g_e(Y; \underline{X}) \quad (24)$$

of the quadratic and exponential function

$$\begin{aligned} g_q(Y; \underline{X}) &\equiv Y(Y - \underline{X}) + \underline{X} \\ g_e(Y; \underline{X}) &\equiv \frac{\underline{X}}{\underline{X}+1} \mathrm{e}^{Y}. \end{aligned} \quad (25)$$

over $Y \in [R, \Sigma - \underline{X}(M-1)]$.

Proof: The derivative of $\underline{T}^{\uparrow}(Y)$ with respect to Y is

$$\frac{\partial}{\partial Y} \underline{T}^{\uparrow}(Y) = h_1(Y; \underline{X}) h_2(Y; \underline{X}, R) \quad (26)$$

where

$$\begin{aligned} h_1(Y; \underline{X}) &\equiv -(\underline{X}+1)Y^2 + \underline{X}(\underline{X}+1)(Y-1) + \underline{X}\mathrm{e}^{Y} \\ h_2(Y; \underline{X}, R) &\equiv \frac{\mathrm{e}^{-Y}(R-\underline{X})}{(1+\underline{X})(Y-\underline{X})^2}. \end{aligned} \quad (27)$$

As $Y \geq R > \underline{X}$, it follows that $h_2(Y; \underline{X}, R) > 0$, and as such the roots of $\frac{\partial}{\partial Y}\underline{T}^{\uparrow}(Y)$, if any, are the roots of $h_1(Y; \underline{X})$. Rearrangement of $h_1(Y; \underline{X}) = 0$ yields (24). ■

The motivation behind the change in variable from K to Y (22) is that the stationary points of $\underline{T}^{\uparrow}(Y)$ depend solely on the value of $\underline{X}$, not on (Σ, M). This limited dependence is not as apparent when working directly with $\underline{T}^{\uparrow}(K)$.

Returning to (24), observe both $g_q(Y; \underline{X})$ and $g_e(Y; \underline{X})$ are convex increasing functions, as illustrated in Fig. 2. The figure demonstrates that up to two intersections are possible for $\underline{X} < 0.774$ but no intersections are possible for $\underline{X} > 0.774$.

Recall, the intersections of (g_q, g_e), if any, must lie within the image of Y, i.e., $[R, \Sigma - \underline{X}(M-1)]$. Future work will seek to characterize regimes of the parameters $(\Sigma, M, \underline{X})$ that hold common extremal values for quasi-uniform points K.

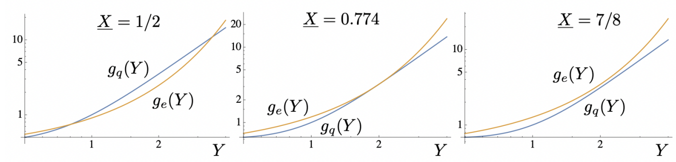

Fig. 2. Functions $g_q(Y; \underline{X})$ and $g_e(Y; \underline{X})$ from Prop. 2 for $\underline{X}$ equal to $1/2$ (left), 0.774 (middle), and $7/8$ (right). Their intersections over $Y \in [R, \Sigma - M(\underline{X}-1)]$, if any, determine the stationary points Y which in turn establish the optimal quasi-uniform feasible point, K. Up to two intersections are possible for $\underline{X} < 0.774$ but no intersections are possible for $\underline{X} > 0.774$.

V. Throughput Lower Bound Minimization with $M = 2$ Channels

Restriction to $M = 2$ channels is of interest because the resulting low dimension of the feasible set $\mathcal{A}$ allows for visualization and intuition that is more difficult for $M > 2$. This section is organized as follows: §V-A specializes the feasible set and objective to $M = 2$, §V-B introduces balanced and imbalanced allocations, §V-C offers necessary conditions for optimality, and §V-D gives two series approximations of the balance to imbalance throughput difference.

A. Problem definition

Specializing the feasible set $\mathcal{A}$ in Def. 1 to $M = 2$ yields:

Definition 3. *For $M = 2$ and $\bar{X} = \infty$ and given system parameters $(N, \Sigma, \underline{X})$ obeying (12), the feasible set $\mathcal{A}$ holds all feasible (n, μ) values:*

$$\begin{aligned} \mathcal{A} \;=\; &\{((n_1, n_2), (\mu_1, \mu_2)) : \\ &n_1 + n_2 = N \\ &n_1\mu_1 + n_2\mu_2 = \Sigma \\ &\underline{X} \leq \min(\mu_1, \mu_2) \\ &1 \leq \min(n_1, n_2)\}. \end{aligned} \quad (28)$$

Note that selection of (n_1, μ_1) determines (n_2, μ_2), i.e., there are two degrees of freedom, and as such the feasible set is viewed on the (n_1, μ_1) plane.

Next, specializing the per-channel throughput lower bound, $\underline{T}(n, \mu)$ in (10) to $M = 2$ yields

$$\underline{T}(n_1, \mu_1) = \frac{1}{2} \left[\underline{\tau}(n_1, \mu_1) + \underline{\tau}\left(N - n_1, \frac{\Sigma - n_1\mu_1}{N - n_1} \right) \right], \quad (29)$$

where notation $\underline{T}(n_1, \mu_1)$ replacing $\underline{T}(n, \mu)$ reflects the fact that, for $M = 2$, a point (n, μ) in $\mathcal{A}$ is determined by (n_1, μ_1).

The optimization problem $\underline{\mathrm{P}}$ in (15) for $M = 2$ becomes

$$\begin{aligned} \underline{\mathrm{P}} : \quad & \min_{(n_1, \mu_1) \in \mathcal{A}} \underline{T}(n_1, \mu_1) \\ \text{s.t.} \quad h_1 :& \quad 1 - n_1 \leq 0 \\ h_2 :& \quad n_1 - (N-1) \leq 0 \\ h_3 :& \quad \underline{X} - \mu_1 \leq 0 \\ h_4 :& \quad n_1(\mu_1 - \underline{X}) - (\Sigma - \underline{X}N) \leq 0 \end{aligned} \quad (30)$$

Constraints h_1 and h_2 capture $\min(n_1, n_2) \geq 1$, while constraints h_3 and h_4 capture $\min(\mu_1, \mu_2) \geq \underline{X}$, where h_4 may be

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rewritten as $\mu_2 \geq \utilde{X}$ for $\mu_2 = (\Sigma - n_1\mu_1)/(N-n_1)$. Observe the feasible set defined by $h_1, \ldots, h_4$ is not polyhedral, on account of the dependence of h_4 on the product $n_1\mu_1$; in fact the constraint may also be written as

$$\mu_1(n_1) \leq \utilde{X} + \frac{\Sigma - \utilde{X}N}{n_1}. \tag{31}$$

B. Balanced and imbalanced allocations

Channel allocations that load balanced and maximally imbalanced, defined below, are of particular interest.

Definition 4 (Balanced and imbalanced allocations)**.** *The (load) balanced allocation splits users and load equally across the two channels:*

$$(n_1, n_2) = \left(\frac{N}{2}, \frac{N}{2}\right), \quad (\mu_1, \mu_2) = \left(\frac{\Sigma}{N}, \frac{\Sigma}{N}\right). \tag{32}$$

The (maximally) imbalanced allocation puts one user and minimum load on one channel, and the remaining users and allocation on the other channel:

$$(n_1, n_2) = (1, N-1), \quad (\mu_1, \mu_2) = \left(\utilde{X}, \frac{\Sigma - \utilde{X}}{N-1}\right). \tag{33}$$

The balanced and imbalanced allocations correspond to $K = 0$ and $K = M - 1$, respectively, in Def. 2. There is no loss in generality with the imbalanced allocation in placing the single user on channel 1, due to the channel symmetry.

The balanced and imbalanced throughputs are

$$\begin{aligned} \utilde{T}\left(\frac{N}{2}, \frac{\Sigma}{N}\right) &= \frac{\Sigma}{2(1+\frac{\Sigma}{N})^{\frac{N}{2}}} \\ \utilde{T}(1, \utilde{X}) &= \frac{1}{2}\left[\frac{\utilde{X}}{1+\utilde{X}} + \frac{\Sigma - \utilde{X}}{(1+\frac{\Sigma - \utilde{X}}{N-1})^{N-1}}\right]. \end{aligned} \tag{34}$$

Definition 5 (Imbalance to balance throughput difference)**.** *The (imbalanced to balanced) throughput difference δ_{ib}, viewed as a function of $\utilde{X}$, is defined as*

$$\begin{aligned} \delta_{ib}(\utilde{X}) &\equiv \utilde{T}(1, \utilde{X}) - \utilde{T}\left(\frac{N}{2}, \frac{\Sigma}{N}\right) \\ &= \frac{1}{2}\left[\frac{\utilde{X}}{1+\utilde{X}} + \frac{\Sigma - \utilde{X}}{(1+\frac{\Sigma-\utilde{X}}{N-1})^{N-1}} - \frac{\Sigma}{(1+\frac{\Sigma}{N})^{\frac{N}{2}}}\right] \end{aligned} \tag{35}$$

Observe the equivalence:

$$\begin{aligned} \delta_{ib}(\utilde{X}) \leq 0 \quad &\Leftrightarrow \quad \utilde{T}(1, \utilde{X}) \leq \utilde{T}\left(\frac{N}{2}, \frac{\Sigma}{N}\right) \\ \delta_{ib}(\utilde{X}) \geq 0 \quad &\Leftrightarrow \quad \utilde{T}(1, \utilde{X}) \geq \utilde{T}\left(\frac{N}{2}, \frac{\Sigma}{N}\right) \end{aligned} \tag{36}$$

C. Necessary conditions for optimality of balanced and imbalanced allocations

Proposition 3. *For $M = 2$ and $\bar{X} = \infty$ and given system parameters $(N, \Sigma, \utilde{X})$ obeying (12):*

- *The balanced allocation in Def. 4 is a stationary point of Problem $\utilde{\mathrm{P}}$ (30) if*

$$\Sigma \geq N\left(\mathrm{e}^{W\left(\frac{2}{N}\right)} - 1\right), \tag{37}$$

where W denotes the Lambert W function, satisfying $W(x)\mathrm{e}^{W(x)} = x$.

- *The imbalanced allocation in Def. 4 is a stationary point of Problem $\utilde{\mathrm{P}}$ in the limit as $\utilde{X} \downarrow 0$.*

The proof is in the Appendix.

Proposition 4. *For $M = 2$ and $\bar{X} = \infty$ and given system parameters (N, Σ) obeying (12), the imbalanced allocation has smaller expected throughput than the balanced allocation in the limit as $\utilde{X} \downarrow 0$.*

Proof: The function $f(n; x) \equiv (1 + x/n)^n$ is increasing in n. As such, from Def. 5,

$$\frac{2}{\Sigma}\lim_{\utilde{X}\to 0} \delta_{ib}(\utilde{X}) = \frac{1}{(1+\frac{\Sigma}{N-1})^{N-1}} - \frac{1}{(1+\frac{\Sigma}{N})^{\frac{N}{2}}} < 0. \tag{38}$$

From (36), $\lim_{\utilde{X}\downarrow 0} \utilde{T}(1, \utilde{X}) \leq \utilde{T}(N/2, \Sigma/N)$. ∎

As $\utilde{X} \downarrow 0$, the imbalanced allocation becomes a single channel ($M = 1$) system, and as such its lower throughput relative to a balanced ($M = 2$) system in Prop. 4 reflects the increased throughput achievable with an additional channel.

D. Series approximation of the balance to imbalance throughput difference

Consider the (N, Σ) plane, and define the regions:

$$\begin{aligned} \mathcal{R}_i(\utilde{X}) &\equiv \{(N, \Sigma) : \delta_{ib}(\utilde{X}) < 0\} \\ \mathcal{R}_b(\utilde{X}) &\equiv \{(N, \Sigma) : \delta_{ib}(\utilde{X}) > 0\} \end{aligned} \tag{39}$$

where regions $\mathcal{R}_i(\utilde{X}), \mathcal{R}_b(\utilde{X})$ have boundary

$$\Delta_{ib}(\utilde{X}) \equiv \{(N, \Sigma) : \delta_{ib}(\utilde{X}) = 0\}. \tag{40}$$

As solution of $\delta_{ib}(\utilde{X}) = 0$ is difficult, the first two Taylor series approximations in $\utilde{X}$ around $\utilde{X} = X$ are employed:

$$\begin{aligned} \bar{\delta}_{ib}(\utilde{X}) &= \delta_{ib}(X) + \delta'_{ib}(X)(\utilde{X} - X) + R_1(X) \\ \utilde{\delta}_{ib}(\utilde{X}) &= \delta_{ib}(X) + \delta'_{ib}(X)(\utilde{X} - X) \\ &\quad + \frac{1}{2}\delta''_{ib}(X)(\utilde{X} - X)^2 + R_2(X) \end{aligned} \tag{41}$$

The first two derivatives of δ'_{ib} (Def. 5) with respect to $\utilde{X}$ are

$$\begin{aligned} \delta'_{ib}(\utilde{X}) &= \frac{1}{2}\left[\frac{1}{(1+\utilde{X})^2} + \frac{(N-2)(\Sigma - \utilde{X}) - (N-1)}{(N-1)(1+\frac{\Sigma-\utilde{X}}{N-1})^N}\right] \\ \delta''_{ib}(\utilde{X}) &= \frac{1}{2}\left[\frac{-2}{(1+\utilde{X})^3} + \frac{(N-2)(\Sigma - \utilde{X}) - 2(N-1)}{(N-1+\Sigma-\utilde{X})(1+\frac{\Sigma-\utilde{X}}{N-1})^N}\right] \end{aligned} \tag{42}$$

and the Lagrange error for the k^{th}-order Taylor series approximation is defined as

$$R_k(X) \equiv \frac{\delta_{ib}^{(k+1)}(X)}{(k+1)!}(\utilde{X} - X)^{k+1}. \tag{43}$$

Proposition 5. *The first ($\bar{\delta}_{ib}$) and second ($\utilde{\delta}_{ib}$) Taylor series approximations (41) upper and lower bound the throughput difference $\delta_{ib}(\utilde{X})$ (Def. 5):*

$$\utilde{\delta}_{ib}(\utilde{X}) \overset{(a)}{\leq} \delta_{ib}(\utilde{X}) \overset{(b)}{\leq} \bar{\delta}_{ib}(\utilde{X}), \tag{44}$$

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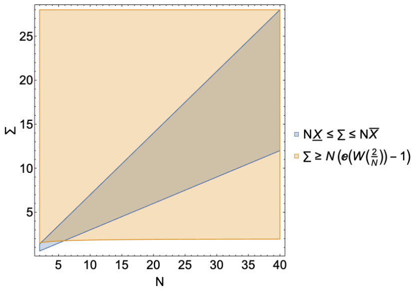

Fig. 3. Two regions on the (N, Σ) plane: $i)$ The system parameter requirement $N\underline{X} \leq \Sigma \leq N\bar{X}$ (12) for $(N, \underline{X}, \bar{X}) = (40, 0.3, 0.7)$ (blue); $ii)$ Region satisfying (37) in the necessary conditions from Prop. 3 for balanced allocations to be stationary solutions of optimization problem $\underline{\mathrm{P}}$ (30, gold).

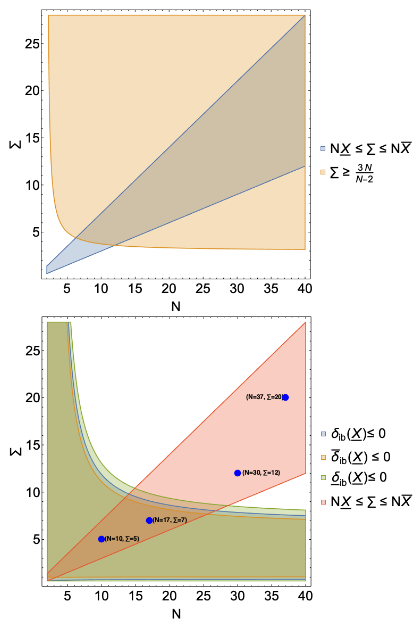

Fig. 4. **Top:** Two regions on the (N, Σ) plane: $i)$ The system parameter requirement $N\underline{X} \leq \Sigma \leq N\bar{X}$ (12) for $(N, \underline{X}, \bar{X}) = (40, 0.3, 0.7)$ (blue); $ii)$ the region satisfying the condition $\Sigma \geq 3N/(N-2)$ required for the ordering of the two series approximations (Prop. 5, gold). **Bottom:** Three regions on the (N, Σ) plane: $i)$ The system parameter requirement $N\underline{X} \leq \Sigma \leq N\bar{X}$ (12) for $(N, \underline{X}, \bar{X}) = (40, 0.3, 0.7)$ (red);$ii)$ the regions where balanced ($\mathcal{R}_b(\underline{X})$, light green) and imbalanced ($\mathcal{R}_i(\underline{X})$, dark green) allocations have smaller throughput (39). The four blue points are the four (N, Σ) pairs used in Fig. 5 and 6. Also shown are the linear and quadratic lower and upper bounds on the boundary $\Delta_{ib}(\underline{X})$ (40) from Prop. 5 (with the series taken around the point $X = 1/10$)

where (a) *holds provided* $X \leq \underline{X}$ *and* (b) *holds provided* $X \leq \underline{X}$ *and* $\Sigma \geq 3N/(N-2)$. *Under these conditions, solutions of the corresponding linear and quadratic equations*

$$\underline{\delta}_{ib}(X) = 0, \quad \bar{\delta}_{ib}(\underline{X}) = 0 \tag{45}$$

yields lower and upper bounds on the boundary $\Delta_{ib}(\underline{X})$ *(40), defined by the nonlinear equation* $\delta_{ib}(\underline{X}) = 0$.

The proof is in the Appendix.

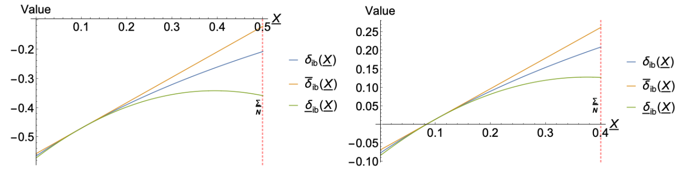

Fig. 5. Imbalance to balance throughput difference function $\delta_{ib}(\underline{X})$ (Def. 5) vs. $\underline{X}$ and its linear ($\underline{\delta}_{ib}$) and quadratic ($\bar{\delta}_{ib}$) bounds from Prop. 5 for (N, Σ) pairs highlighted in blue in Fig. 4: $(10, 5)$ for left and $(30, 12)$ for right.

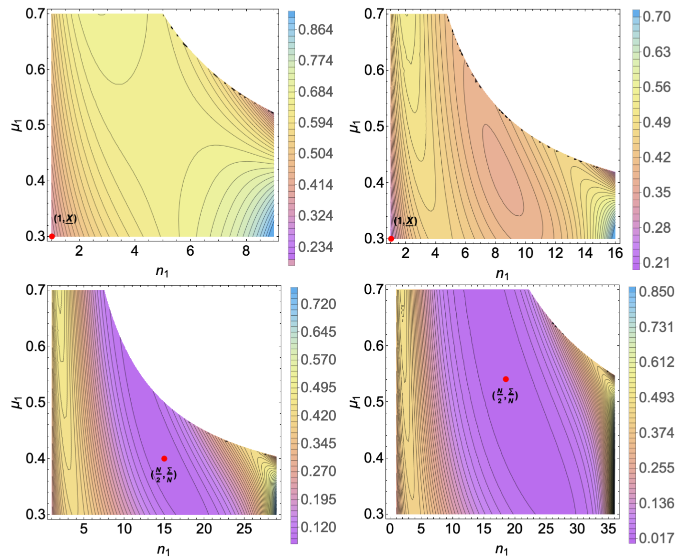

Fig. 6. Contour plots of $\underline{T}(n_1, \mu_1)$ (29) on the (n_1, μ_1) plane for the four (N, Σ) pairs highlighted in blue in Fig. 4: $(10, 5)$ (top left), $(17, 7)$ (top right), $(30, 12)$ (bottom left), and $(37, 20)$ (bottom right). The imbalanced allocation $(1, \underline{X})$ achieves minimum throughput for the top figures, while the balanced allocation $(N/2, \Sigma/N)$ achieves minimum throughput for the bottom figures (red points).

VI. Numerical Results

Numerical results are shown in Figures 3 through 6. All plots in this section are for the $M = 2$ analysis in §V. Although the analysis in the preceding sections assumed $\bar{X} = \infty$, in this section a value of $\bar{X} = 0.7$ is used. Other default values include $\underline{X} = 0.3$ and $N = 40$.

Fig. 3 shows the region under which the balanced allocation is a stationary solution of optimization problem $\underline{\mathrm{P}}$ (30, gold) as discussed in Prop. 3 while the blue region shows the set of feasible parameters $N\underline{X} \leq \Sigma \leq N\bar{X}$ (12).

Fig. 4 top shows the set of feasible parameters $N\underline{X} \leq \Sigma \leq N\bar{X}$ in blue along with the region satisfying the condition $\Sigma \geq \frac{3N}{N-2}$ for $\delta_{ib}(\underline{X})$ to be bounded by it linear and quadratic approximations as discussed in Prop. 5 in gold. The bottom plot shows the boundary $\Delta_{ib}(\underline{X})$ (40), shown as the blue curve, that delineates the regions $\mathcal{R}_i(\underline{X})$ and $\mathcal{R}_b(\underline{X})$ (39) in which the imbalanced and balanced throughputs have lower throughput, respectively. For any (N, Σ) pair in the region $\mathcal{R}_i(\underline{X})$, the imbalanced allocation is the global extrema while and any (N, Σ) pair in $\mathcal{R}_b(\underline{X})$ signifies that balanced allocation is the global extrema. It also shows $i)$ the four

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(N, Σ) pairs used in Fig. 5 and 6 and ii) the linear (upper) and quadratic (lower) bounds on the boundary $\Delta_{ib}(\underline{X})$ (40) from Prop. 5. The four (N, Σ) pairs are chosen such that 2 of them lie in $\mathcal{R}_i(\underline{X})$ and other two in $\mathcal{R}_b(\underline{X})$ in order to show the existence of 2 distinct global extremas.

Fig. 5 shows the imbalance to balance throughput difference function $\delta_{ib}(\underline{X})$ (Def. 5) vs. $\underline{X}$ and its linear ($\underline{\delta}_{ib}$) and quadratic ($\bar{\delta}_{ib}$) bounds from Prop. 5: the absence of a root in the top figure means $\delta_{ib}(\underline{X}) < 0$ for all feasible $\underline{X}$, while a root near $\underline{X} = 0.08$ is evident in the bottom figure.

Fig. 6 shows four contour plots of $\underline{T}(n_1, \mu_1)$ on the (n_1, μ_1) plane, for the four (N, Σ) points shown in Fig. 4. The top (bottom) two plots show the imbalanced (balanced) allocation minimizes throughput over the feasible set, respectively.

VII. Conclusion

The performance of multi-channel random access systems serving heterogeneous users depends upon the user to channel assignment, but the corresponding combinatorial optimization problems (13) are difficult to solve. This motivates the introduction of nonlinear optimization problems that provide lower and upper bounds (15). Focusing on the lower bound, $\underline{\mathsf{P}}$, this paper analyzed the problem in two regimes: i) the many small users regime ($N \uparrow \infty$) in §IV, and ii) the two channel ($M = 2$) case in §V. The focus is on throughput comparison among quasi-uniform allocations (Def. 2) or between balanced and imbalanced allocations (Def. 4), where the results are in terms of solutions of nonlinear equations (Prop. 2) or Taylor series approximations of nonlinear equations (Prop. 5).

Ongoing and future work will continue to investigate the optimization problems $\underline{\mathsf{P}}, \bar{\mathsf{P}}$ (15), with the focus on characterizing the optimal allocations (n, μ) as a function of the system parameters $(M, N, \Sigma, \underline{X}, \bar{X})$.

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Appendix

Only proof sketches are included due to space constraints.

A. Proof of Prop. 3

Proof: The Lagrangian for (30) is used to establish KKT conditions, i.e., primal and dual feasibility and complementary slackness. For the balanced point $(n_1, \mu_1) = (N/2, \Sigma/N)$, the KKT conditions require $\frac{\partial}{\partial n_1} \underline{T}(n_1, \mu_1) \leq 0$. An analysis of this inequality, omitted here, yields the condition (37). For the imbalanced point $(n_1, \mu_1) = (1, \underline{X})$, taking the limit $\underline{X} \downarrow 0$ yields KKT conditions that analysis, omitted here, shows wil always hold, ensuring this point is asymptotically stationary. ■

B. Proof of Prop. 5

Proof: Properties of the balance to imbalance throughput difference function $\delta_{ib}(\underline{X})$ (Def. 5) include the following inequalities on the first two derivatives (42):

- $\delta_{ib}(\underline{X})$ is increasing in $\underline{X}$, i.e., $\delta'_{ib}(\underline{X}) > 0$, over $\underline{X} \in (0, \frac{\Sigma}{N})$.
- $\delta_{ib}(\underline{X})$ is concave in $\underline{X}$, i.e., $\delta''_{ib}(\underline{X}) \leq 0$, over $\underline{X} \in (0, \frac{\Sigma}{N})$ for any $N \geq 3$.

The proof of these two properties is omitted here, due to space limitations. Using the Lagrange error $R_k(X)$ (43), the two approximations (41) are shown to obey the ordering asserted in Prop. 5 as follows. First, the functions $(\delta_{ib}, \bar{\delta}_{ib}, R_1)$ are related as

$$\delta_{ib}(\underline{X}) = \bar{\delta}_{ib}(\underline{X}) + R_1(X) \tag{46}$$

where

$$R_1(X) = \frac{\delta''_{ib}(X)}{2}(\underline{X} - X)^2. \tag{47}$$

As $\delta'_{ib}(\underline{X}) > 0$ and $\delta''_{ib}(\underline{X}) \leq 0$, it follows that $\delta_{ib}(\underline{X}) \leq \bar{\delta}_{ib}(\underline{X})$. Second, the functions $(\delta_{ib}, \underline{\delta}_{ib}, R_2)$ are related as

$$\delta_{ib}(\underline{X}) = \underline{\delta}_{ib}(\underline{X}) + R_2(X) \tag{48}$$

where

$$R_2(X) = \frac{\delta^{(3)}_{ib}(X)}{3!}(\underline{X} - X)^3 \tag{49}$$

requires the third derivative $\delta^{(3)}_{ib}(X)$:

$$\delta^{(3)}_{ib}(X) = \frac{6}{(1+X)^4} + \frac{N((N-2)(\Sigma - X) - 3(N-1))}{((\Sigma - X) + (N-1))^2 (1 + \frac{\Sigma - X}{N-1})^N} \tag{50}$$

Analysis, omitted here, shows $R_2(X) \geq 0$ for $X < \frac{\Sigma}{N}$ and $\Sigma \geq 3N/(N-2)$. It follows that $\delta_{ib}(\underline{X}) \geq \underline{\delta}_{ib}(\underline{X})$. ■

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