Sampling for Remote Estimation through Queues: Age of Information and Beyond

Tasmeen Zaman Ornee and Yin Sun
Dept. of ECE, Auburn University, Auburn, AL

Abstract— **Recently, a connection between the age of information and remote estimation error was found in a sampling problem of Wiener processes: If the sampler has no knowledge of the signal being sampled, the optimal sampling strategy is to minimize the age of information; however, by exploiting causal knowledge of the signal values, it is possible to achieve a smaller estimation error. In this paper, we generalize the previous study by investigating a problem of sampling a stationary Gauss-Markov process named the Ornstein-Uhlenbeck (OU) process, where we aim to find useful insights for solving the problems of sampling more general signals. The optimal sampling problem is formulated as a constrained continuous-time Markov decision process (MDP) with an uncountable state space. We provide an exact solution to this MDP: The optimal sampling policy is a threshold policy on *instantaneous estimation error* and the threshold is found. Further, if the sampler has no knowledge of the OU process, the optimal sampling problem reduces to an MDP for minimizing a *nonlinear* age of information metric and the age-optimal sampling policy is a threshold policy on *expected estimation error* and the threshold is found. In both problems, the optimal sampling policies can be computed by bisection search, and the curse of dimensionality is circumvented. These results hold for (i) general service time distributions of the queueing server and (ii) sampling problems both with and without a sampling rate constraint. Numerical results are provided to compare different sampling policies.**

I. Introduction

Timely updates of the system state are of significant importance for state estimation and decision making in networked control and cyber-physical systems, such as UAV navigation, robotics control, mobility tracking, and environment monitoring systems. To evaluate the freshness of state updates, the concept of *Age of Information*, or simply *age*, was introduced to measure the timeliness of state samples received from a remote transmitter [1]–[3]. Let U_t be the generation time of the freshest received state sample at time t. The age of information, as a function of t, is defined as $\Delta_t = t - U_t$, which is the time difference between the freshest samples available at the transmitter and receiver.

Recently, the age of information concept has received significant attention, because of the extensive applications of state updates among systems connected over communication networks. The states of many systems, such as UAV mobility trajectory and sensor measurements, are in the form of a signal X_t, that may change slowly at some time and vary more dynamically later. Hence, the time difference described by the age $\Delta_t = t - U_t$ only partially characterize the variation $X_t - X_{U_t}$ of the system state, and the state update policy that minimizes the age of information does not minimize the state estimation error. This result was first shown in [4], [5], where a sampling problem of Wiener processes was solved and the optimal sampling policy was shown to have an intuitive structure. As the results therein hold only for non-stationary signals that can be modeled as a Wiener process, one would wonder how to, and whether it is possible to, extend [4], [5] for handling more general signal models.

In this paper, we generalize [4], [5] by exploring a problem of sampling an Ornstein-Uhlenbeck (OU) process X_t. From the obtained results, we hope to find useful structural properties of the optimal sampler design that can be potentially applied to more general signal models. The OU process X_t is the continuous-time analogue of the well-known first-order autoregressive process, i.e., AR(1) process. The OU process is defined as the solution to the stochastic differential equation (SDE) [6], [7]

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t, \tag{1}$$

where μ, $\theta > 0$, and $\sigma > 0$ are parameters and W_t represents a Wiener process. It is the only nontrivial continuous-time process that is stationary, Gaussian, and Markovian [7]. Examples of first-order systems that can be described as the OU process include interest rates, currency exchange rates, and commodity prices (with modifications) [8], control systems such as node mobility in mobile ad-hoc networks, robotic swarms, and UAV systems [9], [10], and physical processes such as the transfer of liquids or gases in and out of a tank [11].

As shown in Fig. 1, samples of an OU process are forwarded to a remote estimator through a channel in a first-come, first-served fashion. The samples experience *i.i.d.* random transmission times over the channel, which is caused by random sample size, channel fading, interference, congestions, and etc. For examples, UAVs flying close to WiFi access points may suffer from long communication delay and instability issues, because they receive strong interference from the WiFi access points [12]. We assume that at each time only one sample can be served by the channel. The samples that are waiting to be sent are stored in a queue at the transmitter. Hence, the channel is modeled as a FIFO queue with *i.i.d.* service times. The service time distributions considered in this paper are quite general: they are only required to have a finite mean. This queueing model is helpful to analyze the robustness of remote estimation systems with occasionally long transmission times.

The estimator utilizes causally received samples to construct an estimate $\hat{X}_t$ of the real-time signal value X_t. The quality of remote estimation is measured by the time-average mean-

This work was supported in part by NSF grant CCF-1813050 and ONR grant N00014-17-1-2417.

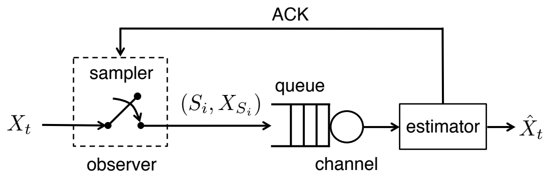

Fig. 1: System model.

squared estimation error, i.e.,

$$\mathsf{mse} = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (X_t - \hat{X}_t)^2 dt\right]. \tag{2}$$

Our goal is to find the optimal causal sampling policy that minimizes mse by choosing the sampling times subject to a maximum sampling rate constraint. In practice, the cost (e.g., energy, CPU cycle, storage) for state updates increases with the average sampling rate. Hence, we are striking to find the optimum tradeoff between estimation error and update cost. In addition, the unconstrained problem will also be solved. The contributions of this paper are summarized as follows:

- The optimal sampling problem for minimize the mse under a sampling rate constraint is formulated as a constrained continuous-time Markov decision process (MDP) with an uncountable state space. Because of the curse of dimensionality, such problems are often lack of low-complexity solutions that are arbitrarily accurate. However, we were able to solve this MDP exactly: The optimal sampling policy is proven to be a threshold policy on *instantaneous* estimation error, where the threshold is a non-linear function $v(\beta)$ of a parameter β. The value of β is equal to the summation of the optimal objective value of the MDP and the optimal Lagrangian dual variable associated to the sampling rate constraint. If there is no sampling rate constraint, the Lagrangian dual variable is zero and hence β is exactly the optimal objective value. Among the technical tools developed to prove this result is a free boundary method [13], [14] for finding the optimal stopping time of the OU process.
- The optimal sampler design of Wiener process in [4], [5] is a limiting case of the above result. By comparing the optimal sampling policies of OU process and Wiener process, we find that the threshold function $v(\beta)$ changes according to the signal model, where the parameter β is determined in the same way for both signal models.
- Further, we consider a class of signal-ignorant sampling policies, where the sampling times are determined without using knowledge of the observed OU process. The optimal signal-ignorant sampling problem is equivalent to an MDP for minimizing the time-average of a nonlinear age function $p(\Delta_t)$, which has been solved recently in [15]. The age-optimal sampling policy is a threshold policy on *expected* estimation error, where the threshold function is simply $v(\beta) = \beta$ and the parameter β is determined in the same way as above.
- The above results hold for (i) general service time distributions with a finite mean and (ii) sampling problems both with and without a sampling rate constraint. Numerical results suggest that the optimal sampling policy is better than zero-wait sampling and the classic uniform sampling.

One interesting observation from these results is that the threshold function $v(\beta)$ varies with respect to the signal model and sampling problem, but the parameter β is determined in the same way.

A. Related Work

The results in this paper are tightly related to recent studies on the age of information Δ_t, e.g., [1], [15]–[32], which does not have a signal model. The average age and average peak age have been analyzed for various queueing systems in, e.g., [1], [19], [21], [22]. The optimality of the Last-Come, First-Served (LCFS) policy, or more generally the Last-Generated, First-Served (LGFS) policy, was established for various queueing system models in [25]–[29]. Optimal sampling policies for minimizing non-linear age functions were developed in [15]–[17], [20]. Age-optimal transmission scheduling of wireless networks were investigated in, e.g., [23], [24], [30]–[34].

On the other hand, this paper is also a contribution to the area of remote estimation, e.g., [11], [35]–[40], by adding a queue between the sampler and estimator. In [36], [38], optimal sampling of Wiener processes was studied, where the transmission time from the sampler to the estimator is zero. Optimal sampling of OU processes was also considered in [36], which is solved by discretizing time and using dynamic programming to solve the discrete-time optimal stopping problems. However, our optimal sampler of OU processes is obtained analytically. Remote estimation over several different channel models was recently studied in, e.g., [39], [40]. In [11], [35]–[40], the optimal sampling policies were proven to be threshold policies. Because of the queueing model, our optimal sampling policy has a different structure from those in [11], [35]–[40]. Specifically, in our optimal sampling policy, sampling is suspended when the server is busy and is reactivated once the server becomes idle. The optimal sampling policy for Wiener processes in [4], [5] is a limiting case of ours.

II. Model and Formulation

A. System Model

We consider the remote estimation system illustrated in Fig. 1, where an observer takes samples from an OU process X_t and forwards the samples to an estimator through a communication channel. The channel is modeled as a single-server FIFO queue with *i.i.d.* service times. The system starts to operate at time $t = 0$. The i-th sample is generated at time S_i and is delivered to the estimator at time D_i with a service time Y_i, which satisfy $S_i \le S_{i+1}$, $S_i + Y_i \le D_i$, $D_i + Y_{i+1} \le D_{i+1}$, and $0 < \mathbb{E}[Y_i] < \infty$ for all i. Each sample packet (S_i, X_{S_i}) contains the sampling time S_i and the sample value X_{S_i}. Let $U_t = \max\{S_i : D_i \le t\}$ be the sampling time of the latest received sample at time t. The *age of information*, or simply *age*, at time t is defined as [1]

$$\Delta_t = t - U_t = t - \max\{S_i : D_i \le t\}, \tag{3}$$

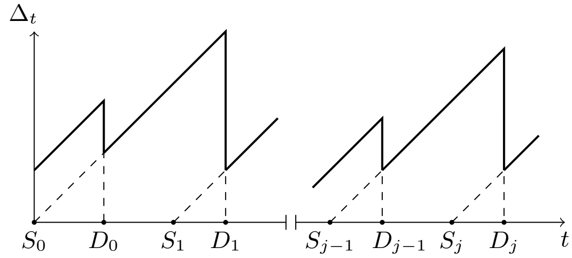

Fig. 2: Evolution of the age Δ_t over time.

which is shown in Fig. 2. Because $D_i \leq D_{i+1}$, Δ_t can be also expressed as

$$\Delta_t = t - S_i, \text{ if } t \in [D_i, D_{i+1}), \ i = 0, 1, 2, \ldots \quad (4)$$

The initial state of the system is assumed to satisfy $S_0 = 0$, $D_0 = Y_0$, X_0 and Δ_0 are finite constants. The parameters μ, θ, and σ in (1) are known at both the sampler and estimator.

Let $I_t \in \{0, 1\}$ represent the idle/busy state of the server at time t. We assume that whenever a sample is delivered, an acknowledgement is sent back to the sampler with zero delay. By this, the idle/busy state I_t of the server is known at the sampler. Therefore, the information that is available at the sampler at time t can be expressed as $\{X_s, I_s : 0 \leq s \leq t\}$.

B. Sampling Policies

In causal sampling policies, each sampling time S_i is chosen by using the up-to-date information available at the sampler. To characterize this statement precisely, let us define the σ-fields

$$\mathcal{N}_t = \sigma(X_s, I_s : 0 \leq s \leq t), \ \mathcal{N}_t^+ = \cap_{s>t}\mathcal{N}_s. \quad (5)$$

Hence, $\{\mathcal{N}_t^+, t \geq 0\}$ is a *filtration* (i.e., a non-decreasing and right-continuous family of σ-fields) of the information available at the sampler. Each sampling time S_i is a *stopping time* with respect to the filtration $\{\mathcal{N}_t^+, t \geq 0\}$ such that

$$\{S_i \leq t\} \in \mathcal{N}_t^+, \ \forall t \geq 0. \quad (6)$$

Let $\pi = (S_1, S_2, ...)$ represent a sampling policy and Π represent the set of *causal* sampling policies that satisfy two conditions: (i) Each sampling policy $\pi \in \Pi$ satisfies (6) for all i. (ii) The sequence of inter-sampling times $\{T_i = S_{i+1} - S_i, i = 0, 1, \ldots\}$ forms a *regenerative process* [41, Section 6.1]: There exists an increasing sequence $0 \leq k_1 < k_2 < \ldots$ of almost surely finite random integers such that the post-k_j process $\{T_{k_j+i}, i = 0, 1, \ldots\}$ has the same distribution as the post-k_0 process $\{T_{k_0+i}, i = 0, 1, \ldots\}$ and is independent of the pre-k_j process $\{T_i, i = 0, 1, \ldots, k_j - 1\}$; further, we assume that $\mathbb{E}[k_{j+1} - k_j] < \infty$, $\mathbb{E}[S_{k_1}] < \infty$, and $0 < \mathbb{E}[S_{k_{j+1}} - S_{k_j}] < \infty, \ j = 1, 2, \ldots$[1]

[1]We will optimize $\limsup_{T\to\infty} \mathbb{E}[\int_0^T (X_t - \hat{X}_t)^2 dt]/T$, but operationally a nicer criterion is $\limsup_{i\to\infty} \mathbb{E}[\int_0^{D_i} (X_t - \hat{X}_t)^2 dt]/\mathbb{E}[D_i]$. These criteria are corresponding to two definitions of "average cost per unit time" that are widely used in the literature of semi-Markov decision processes. These two criteria are equivalent, if $\{T_1, T_2, \ldots\}$ is a regenerative process, or more generally, if $\{T_1, T_2, \ldots\}$ has only one ergodic class. If not condition is imposed, however, they are different. The interested readers are referred to [42]–[46] for more discussions.

From this, we can obtain that S_i is finite almost surely for all i. We assume that the OU process $\{X_t, t \geq 0\}$ and the service times $\{Y_i, i = 1, 2, \ldots\}$ are mutually independent, and do not change according to the sampling policy.

A sampling policy $\pi \in \Pi$ is said to be *signal-ignorant* (*signal-aware*), if π is (not necessarily) independent of $\{X_t, t \geq 0\}$. Let $\Pi_{\text{signal-ignorant}} \subset \Pi$ denote the set of signal-ignorant sampling policies, defined as

$$\Pi_{\text{signal-ignorant}} = \{\pi \in \Pi : \pi \text{ is independent of } \{X_t, t \geq 0\}\}. \quad (7)$$

C. MMSE Estimator

According to (6), S_i is a finite stopping time. By using [47, Eq. (3)] and the strong Markov property of the OU process [13, Eq. (4.3.27)], X_t is expressed as

$$\begin{aligned} X_t =& X_{S_i} e^{-\theta(t-S_i)} + \mu\big[1 - e^{-\theta(t-S_i)}\big] \\ &+ \frac{\sigma}{\sqrt{2\theta}} e^{-\theta(t-S_i)} W_{e^{2\theta(t-S_i)}-1}, \text{ if } t \in [S_i, \infty). \end{aligned} \quad (8)$$

At any time $t \geq 0$, the estimator uses causally received samples to construct an estimate $\hat{X}_t$ of the real-time signal value X_t. The information available to the estimator consists of two parts: (i) $M_t = \{(S_i, X_{S_i}, D_i) : D_i \leq t\}$, which contains the sampling time S_i, sample value X_{S_i}, and delivery time D_i of the samples that have been delivered by time t and (ii) the fact that no sample has been received after the last delivery time $\max\{D_i : D_i \leq t\}$. Similar to [5], [36], [48], we assume that the estimator neglects the second part of information.[2] Then, as shown in our technical report [49], the minimum mean square error (MMSE) estimator is

$$\begin{aligned} \hat{X}_t = \mathbb{E}[X_t | M_t] =& X_{S_i} e^{-\theta(t-S_i)} + \mu\big[1 - e^{-\theta(t-S_i)}\big], \\ & \text{if } t \in [D_i, D_{i+1}), \ i = 0, 1, 2, \ldots \end{aligned} \quad (9)$$

D. Problem Formulation

The goal of this paper is to find the optimal sampling policy that minimizes the mean-squared estimation error subject to an average sampling-rate constraint, which is formulated as the following problem:

$$\mathsf{mse}_{\text{opt}} = \min_{\pi \in \Pi} \ \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (X_t - \hat{X}_t)^2 dt\right] \quad (10)$$

$$\text{s.t. } \liminf_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^n (S_{i+1} - S_i)\right] \geq \frac{1}{f_{\max}}, \quad (11)$$

where $\mathsf{mse}_{\text{opt}}$ is the optimum value of (10) and $f_{\max}$ is the maximum allowed sampling rate. When $f_{\max} = \infty$, this problem becomes an unconstrained problem.

III. Main Result

A. Signal-aware Sampling

Problem (10) is a constrained continuous-time MDP with a continuous state space. However, we found an exact solution to this problem. To present this solution, let us consider an

[2]In [11], [35]–[40], it was shown that when the sampler and estimator are jointly optimized, the MMSE estimator has the same expression no matter with or without the second part of information. We will consider the joint optimization of the sampler and estimator in our future work.

Algorithm 1 Bisection method for solving (19)

given $l = \mathsf{mse}_{Y_i}$, $u = \mathsf{mse}_\infty$, tolerance $\epsilon > 0$.
repeat
  $\beta := (l+u)/2$.
  $o := \beta - \frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)}(X_t-\hat{X}_t)^2 dt\right]}{\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]}$.
  if $o \geq 0$, $u := \beta$; **else**, $l := \beta$.
until $u - l \leq \epsilon$.
return β.

Algorithm 2 Bisection method for solving (21)

given $l = \mathsf{mse}_{Y_i}$, $u = \mathsf{mse}_\infty$, tolerance $\epsilon > 0$.
repeat
  $\beta := (l+u)/2$.
  $o := \mathbb{E}[D_{i+1}(\beta) - D_i(\beta)]$.
  if $o \geq 1/f_{\max}$, $u := \beta$; **else**, $l := \beta$.
until $u - l \leq \epsilon$.
return β.

OU process O_t with initial state $O_t = 0$ and parameter $\mu = 0$. According to (8), O_t can be expressed as

$$O_t = \frac{\sigma}{\sqrt{2\theta}} e^{-\theta t} W_{e^{2\theta t}-1}. \tag{12}$$

Define

$$\mathsf{mse}_{Y_i} = \mathbb{E}[O_{Y_i}^2] = \frac{\sigma^2}{2\theta}\mathbb{E}[1-e^{-2\theta Y_i}], \tag{13}$$

$$\mathsf{mse}_\infty = \mathbb{E}[O_\infty^2] = \frac{\sigma^2}{2\theta}. \tag{14}$$

In the sequel, we will see that mse_{Y_i} and mse_∞ are the lower and upper bounds of $\mathsf{mse}_{\text{opt}}$, respectively. We will also need to use the function[3]

$$G(x) = \frac{e^{x^2}}{x}\int_0^x e^{-t^2} dt = \frac{e^{x^2}}{x}\frac{\sqrt{\pi}}{2}\operatorname{erf}(x),\ x \in [0,\infty), \tag{15}$$

where $\operatorname{erf}(\cdot)$ is the error function, defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt. \tag{16}$$

Theorem 1. *If the Y_i's are i.i.d. with $0 < \mathbb{E}[Y_i] < \infty$, then $(S_1(\beta), S_2(\beta), \ldots)$ with a parameter β is an optimal solution to* (10)*, where*

$$S_{i+1}(\beta) = \inf\left\{t \geq D_i(\beta) : \left|X_t - \hat{X}_t\right| \geq v(\beta)\right\}, \tag{17}$$

$D_i(\beta) = S_i(\beta) + Y_i$, $v(\beta)$ is defined by

$$v(\beta) = \frac{\sigma}{\sqrt{\theta}} G^{-1}\left(\frac{\mathsf{mse}_\infty - \mathsf{mse}_{Y_i}}{\mathsf{mse}_\infty - \beta}\right), \tag{18}$$

$G^{-1}(\cdot)$ is the inverse function of $G(\cdot)$ in (15)*. The value of $\beta \geq 0$ is determined in two cases: β is the root of*

$$\beta = \frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)}(X_t - \hat{X}_t)^2 dt\right]}{\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)]}, \tag{19}$$

if the root of (19) *satisfies*

$$\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] > 1/f_{\max}; \tag{20}$$

otherwise, β is the root of

$$\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] = 1/f_{\max}. \tag{21}$$

The optimal objective value to (10) *is then given by*

$$\mathsf{mse}_{\text{opt}} = \frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)}(X_t - \hat{X}_t)^2 dt\right]}{\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)]}. \tag{22}$$

The proof of Theorem 1 is explained in Section IV. Due to space limitation, most proofs are relegated to our technical report [49], unless specified otherwise. The optimal sampling policy in Theorem 1 has a nice structure. Specifically, the $(i+1)$-th sample is taken at the earliest time t satisfying two conditions: (i) The i-th sample has already been delivered by time t, i.e., $t \geq D_i(\beta)$, and (ii) the estimation error $|X_t - \hat{X}_t|$ is no smaller than a pre-determined threshold $v(\beta)$, where $v(\cdot)$ is a non-linear function defined in (18). In [49], we have obtained

Lemma 1. *In Theorem 1, it holds that $\mathsf{mse}_{Y_i} \leq \mathsf{mse}_{\text{opt}} \leq \beta \leq \mathsf{mse}_\infty$.*

By Lemma 1, $\frac{\mathsf{mse}_\infty - \mathsf{mse}_{Y_i}}{\mathsf{mse}_\infty - \beta} \geq 1$. Further, it is not hard to show that $G(x)$ is strictly increasing on $[0,\infty)$ and $G(0) = 1$. Hence, the inverse function $G^{-1}(\cdot)$ and the threshold $v(\beta)$ are properly defined and $v(\beta) \geq 0$. We note that the service time distribution affects the optimal sampling policy in (17) and (18) through mse_{Y_i} and β.

The calculation of β falls into two cases: In one case, β can be computed by solving (19) via the bisection search method in Algorithm 1. For this case to occur, the sampling rate constraint (11) needs to be inactive at the β obtained in Algorithm 1. Because $D_i(\beta) = S_i(\beta) + Y_i$, we can obtain $\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] = \mathbb{E}[S_{i+1}(\beta) - S_i(\beta)]$ and hence (20) holds when the sampling rate constraint (11) is inactive. In the other case, β is selected to satisfy the sampling rate constraint (11) with equality, which is implemented by using another bisection method in Algorithm 2. The upper and lower bounds for bisection search in Algorithms 1-2 are chosen based on Lemma 1.

If $f_{\max} = \infty$, because $\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] \geq \mathbb{E}[Y_i] > 0$, (20) is always satisfied and only the first case can happen. By comparing (19) and (22), it follows immediately that

Lemma 2. *If the sampling rate constraint is removed, i.e., $f_{\max} = \infty$, then $\beta = \mathsf{mse}_{\text{opt}}$.*

The calculation of the expectations in Algorithms 1-2 can be greatly simplified by using the following lemma, which is obtained in [49] by using Dynkin's formula [14, Theorem 7.4.1] and the optional stopping theorem.

[3]If $x = 0$, $G(x)$ is defined as its right limit $G(0) = \lim_{x\to 0^+} G(x) = 1$.

Lemma 3. *In Theorem 1, it holds that*

$$\begin{aligned}&\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]\\ =&\mathbb{E}[\max\{R_1(v(\beta))-R_1(O_{Y_i})\},0]+\mathbb{E}[Y_i],\end{aligned} \tag{23}$$

$$\begin{aligned}&\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)}(X_t-\hat{X}_t)^2dt\right]\\ =&\mathbb{E}[\max\{R_2(v(\beta))-R_2(O_{Y_i})\},0]\\ &+\mathsf{mse}_\infty[\mathbb{E}(Y_i)-\gamma]+\mathbb{E}\left[\max\{v^2(\beta),O_{Y_i}^2\}\right]\gamma,\end{aligned} \tag{24}$$

where

$$\gamma=\frac{1}{2\theta}\mathbb{E}[1-e^{-2\theta Y_i}], \tag{25}$$

$$R_1(v)=\frac{v^2}{\sigma^2}\,{}_2F_2\left(1,1;\frac{3}{2},2;\frac{\theta}{\sigma^2}v^2\right), \tag{26}$$

$$R_2(v)=-\frac{v^2}{2\theta}+\frac{v^2}{2\theta}\,{}_2F_2\left(1,1;\frac{3}{2},2;\frac{\theta}{\sigma^2}v^2\right), \tag{27}$$

and ${}_2F_2$ is the generalized hypergeometric function [50].

Because the Y_i's are *i.i.d.*, the expectations in Lemma 3 are functions of β and are irrelevant of i. One can improve the accuracy of the solution in Algorithms 1-2 by (i) reducing the tolerance ϵ and (ii) increasing the number of Monte Carlo realizations for computing the expectations. Such a low-complexity solution for solving a constrained continuous-time MDP with a continuous state space is rare.

1) Sampling of Wiener Processes: In the limiting case that $\sigma=1$ and $\theta\to 0$, the OU process X_t in (1) becomes a Wiener process $X_t=W_t$. In this case, the MMSE estimator in (9) is given by

$$\hat{X}_t=W_{S_i},\ \text{if } t\in[D_i,D_{i+1}). \tag{28}$$

As shown in [49], $v(\cdot)$ defined by (18) tends to

$$v(\beta)=\sqrt{3(\beta-\mathbb{E}[Y_i])}. \tag{29}$$

Theorem 2. *If $\sigma=1$, $\theta\to 0$, and the Y_i's are i.i.d. with $0<\mathbb{E}[Y_i]<\infty$, then $(S_1(\beta),S_2(\beta),\ldots)$ with a parameter β is an optimal solution to* (10)*, where*

$$S_{i+1}(\beta)=\inf\left\{t\geq D_i(\beta):\left|X_t-\hat{X}_t\right|\geq\sqrt{3(\beta-\mathbb{E}[Y_i])}\right\}, \tag{30}$$

$D_i(\beta)=S_i(\beta)+Y_i$. The value of $\beta\geq 0$ is determined in two cases: β is the root of

$$\beta=\frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)}(X_t-\hat{X}_t)^2dt\right]}{\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]}, \tag{31}$$

if the root of (31) *satisfies $\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]>1/f_{\max}$; otherwise, β is the root of $\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]=1/f_{\max}$. The optimal objective value to* (10) *is given by*

$$\mathsf{mse}_{\text{opt}}=\frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)}(X_t-\hat{X}_t)^2dt\right]}{\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]}. \tag{32}$$

Theorem 2 is an alternative form of Theorem 1 in [4], [5]. The benefit of the new expression in (30)-(32) is that they allows to character β based on the optimal objective value $\mathsf{mse}_{\text{opt}}$ and the sampling rate constraint (11), in the same way as in Theorem 1, which appears to be more fundamental than the expression in [4], [5].

B. Signal-ignorant Sampling

In signal-ignorant sampling policies, the sampling times S_i are determined based only on the service times Y_i, but not on the observed OU process $\{X_t,t\geq 0\}$.

Lemma 4. *If $\pi\in\Pi_{\text{signal-ignorant}}$, then the mean-squared estimation error of the OU process X_t at time t is*

$$\mathbb{E}\left[(X_t-\hat{X}_t)^2\big|\pi,Y_1,Y_2,\ldots\right]=\frac{\sigma^2}{2\theta}\left(1-e^{-2\theta\Delta_t}\right), \tag{33}$$

which a strictly increasing function $p(\Delta_t)$ of the age Δ_t.

According to Lemma 4 and Fubini's theorem, for every policy $\pi\in\Pi_{\text{signal-ignorant}}$,

$$\mathbb{E}\left[\int_0^T(X_t-\hat{X}_t)^2dt\right]=\mathbb{E}\left[\int_0^T p(\Delta_t)dt\right]. \tag{34}$$

Hence, minimizing the mean-squared estimation error among signal-ignorant sampling policies can be formulated as the following MDP for minimizing the expected time-average of the nonlinear age function $p(\Delta_t)$:

$$\mathsf{mse}_{\text{age-opt}}=\inf_{\pi\in\Pi_{\text{signal-ignorant}}}\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T p(\Delta_t)dt\right] \tag{35}$$

$$\text{s.t.}\quad \liminf_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\sum_{i=1}^n(S_{i+1}-S_i)\right]\geq\frac{1}{f_{\max}}, \tag{36}$$

where $\mathsf{mse}_{\text{age-opt}}$ is the optimal value of (35). By (33), $p(\Delta_t)$ and $\mathsf{mse}_{\text{age-opt}}$ are bounded. Problem (35) is one instance of the problems recently solved in Corollary 3 of [15] for general strictly increasing functions $p(\cdot)$. From this, a solution to (35) is given by

Theorem 3. *If the Y_i's are i.i.d. with $0<\mathbb{E}[Y_i]<\infty$, then $(S_1(\beta),S_2(\beta),\ldots)$ with a parameter β is an optimal solution to* (35)*, where*

$$S_{i+1}(\beta)=\inf\left\{t\geq D_i(\beta):\mathbb{E}[(X_{t+Y_{i+1}}-\hat{X}_{t+Y_{i+1}})^2]\geq\beta\right\}, \tag{37}$$

$D_i(\beta)=S_i(\beta)+Y_i$. The value of $\beta\geq 0$ is determined in two cases: β is the root of

$$\beta=\frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)}(X_t-\hat{X}_t)^2dt\right]}{\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]}, \tag{38}$$

if the root of (38) *satisfies $\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]>1/f_{\max}$; otherwise, β is the root of $\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]=1/f_{\max}$. The optimal objective value to* (35) *is given by*

$$\mathsf{mse}_{\text{age-opt}}=\frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)}(X_t-\hat{X}_t)^2dt\right]}{\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]}. \tag{39}$$

Because $\Pi_{\text{signal-ignorant}}\subset\Pi$, it follows immediately that $\mathsf{mse}_{\text{opt}}\leq\mathsf{mse}_{\text{age-opt}}$.

C. Discussions of the Results

The difference among Theorems 1-3 is only in the expressions (17), (30), (37) of threshold policies. In signal-aware sampling policies (17) and (30), the sampling time is determined by the *instantaneous* estimation error $\left|X_t - \hat{X}_t\right|$, and the threshold function $v(\cdot)$ is determined by the specific signal model. In the signal-ignorant sampling policy (37), the sampling time is determined by the *expected* estimation error $\mathbb{E}[(X_{t+Y_{i+1}} - \hat{X}_{t+Y_{i+1}})^2]$ at time $t + Y_{i+1}$. We note that if $t = S_{i+1}(\beta)$, then $t + Y_{i+1} = S_{i+1}(\beta) + Y_{i+1} = D_{i+1}(\beta)$ is the delivery time of the new sample. Hence, (37) requires that the expected estimation error upon the delivery of the new sample is no less than β. The parameter β in Theorems 1-3 is determined by the optimal objective value and the sampling rate constraint in the same manner. Later on in (42) and in our technical report [49], we have shown that β is exactly equal to the summation of the optimal objective value of the MDP and the optimal Lagrangian dual variable associated to the sampling rate constraint. Finally, it is worth noting that Theorems 1-3 hold for all distributions of the service times Y_i satisfying $0 < \mathbb{E}[Y_i] < \infty$, and for both constrained and unconstrained sampling problems.

IV. A Key Proof Step of Theorem 1: Optimal Per-Sample Stopping Rule

We prove Theorem 1 in four steps: (i) We first show that sampling should be suspended when the server is busy, which can be used to simplify (10). (ii) We use an extended Dinkelbach's method [51] and Lagrangian duality method to decompose the simplified problem into a series of mutually independent per-sample MDP. (iii) We utilize the free boundary method in the optimal stopping theory [13] to solve the per-sample MDPs analytically. (iv) Finally, we use a geometric multiplier method [52] to show that the duality gap is zero. The above proof framework is an extension to that used in [4], [5], [15]. Due to space limitation, we present the most challenging part of the proof in this section, which is step (iii) on finding the analytical solution of the per-sample MDP. The other steps are provided in our technical report [49].

Let us consider an OU process V_t with initial state $V_0 = v$ and parameter $\mu = 0$. Define the σ-fields $\mathcal{F}_t^V = \sigma(V_s : s \in [0,t])$, $\mathcal{F}_t^{V+} = \cap_{r>t}\mathcal{F}_r^V$, and the filtration $\{\mathcal{F}_t^{V+}, t \geq 0\}$ associated to $\{V_t, t \geq 0\}$. Define $\mathfrak{M}_V$ as the set of integrable stopping times of $\{V_t, t \in [0,\infty)\}$, i.e.,

$$\mathfrak{M}_V = \{\tau \geq 0 : \{\tau \leq t\} \in \mathcal{F}_t^{V+}, \mathbb{E}[\tau] < \infty\}. \tag{40}$$

In step (iii), our goal is to solve the following optimal stopping problem for any given initial state $v \in \mathbb{R}$

$$\sup_{\tau\in\mathfrak{M}_V} \mathbb{E}_v\left[-\gamma V_\tau^2 - \int_0^\tau (V_s^2 - \beta)ds\right], \tag{41}$$

where $\mathbb{E}_v[\cdot]$ is the conditional expectation for given initial state $V_0 = v$, $\gamma \geq 0$ is given in (25), and $\beta \geq 0$ is given by

$$\beta = \mathsf{mse}_{\text{opt}} + \lambda, \tag{42}$$

In order to solve (41), we first find a candidate solution to (41) by solving a free boundary problem; then we prove that the free boundary solution is indeed the value function of (41): where the supremum is taken over all stopping times τ of V_t.

1) A candidate solution to (41)*:* Now, we show how to solve (41). The general optimal stopping theory in Chapter I of [13] tells us that the following guess of the stopping time should be optimal for Problem (41):

$$\tau_* = \inf\{t \geq 0 : |V_t| \geq v_*\}, \tag{43}$$

where $v_* \geq 0$ is the optimal stopping threshold to be found. Observe that in this guess, the continuation region $(-v_*, v_*)$ is assumed symmetric around zero since the OU process is symmetric, i.e., the process $\{-V_t, t \geq 0\}$ is also an OU process started at $-v$. Similarly, we may also argue that the value function should be even.

According to [13, Chapter 8], and [14, Chapter 10], the value function and the optimal stopping threshold v_* should satisfy the following free boundary problem:

$$\frac{\sigma^2}{2}H''(v) - \theta v H'(v) = v^2 - \beta, \quad v \in (-v_*, v_*), \tag{44}$$

$$H(\pm v_*) = -\gamma v_*^2, \tag{45}$$

$$H'(\pm v_*) = \mp 2\gamma v_*. \tag{46}$$

In [49], we use the integrating factor method [53, Sec. I.5] to find the general solution to (44) given by

$$H(v) = -\frac{v^2}{2\theta} + \left(\frac{1}{2\theta} - \frac{\beta}{\sigma^2}\right) {}_2F_2\left(1, 1; \frac{3}{2}, 2; \frac{\theta}{\sigma^2}v^2\right) v^2 + C_1 \text{erfi}\left(\frac{\sqrt{\theta}}{\sigma}v\right) + C_2, \qquad v \in (-v_*, v_*), \tag{47}$$

where C_1 and C_2 are constants to be found for satisfying (45)-(46), and $\text{erfi}(x)$ is the imaginary error function, i.e.,

$$\text{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} dt. \tag{48}$$

Because $H(v)$ should be even, $C_1 = 0$. In order to satisfy the boundary condition (45), C_2 is chosen as

$$C_2 = \frac{1}{2\theta}\mathbb{E}\left(e^{-2\theta Y_i}\right) v_*^2 + \left(\frac{1}{2\theta} - \frac{\beta}{\sigma^2}\right) {}_2F_2\left(1, 1; \frac{3}{2}, 2; \frac{\theta}{\sigma^2}v_*^2\right) v_*^2,$$

where we have used (25). By this, we obtain the expression of $H(v)$ in the continuation region $(-v_*, v_*)$. In the stopping region $|v| \geq v_*$, the stopping time in (43) is $\tau_* = 0$ since $|V_0| = |v| \geq v_*$. Hence, if $|v| \geq v_*$, the objective value achieved by the sampling time (43) is

$$\mathbb{E}_v\left[-\gamma v^2 - \int_0^0 (V_s^2 - \beta)ds\right] = -\gamma v^2.$$

By this, we obtain a candidate of the value function for (41):

$$H(v) = \begin{cases} -\frac{v^2}{2\theta} + \left(\frac{1}{2\theta} - \frac{\beta}{\sigma^2}\right) {}_2F_2\left(1, 1; \frac{3}{2}, 2; \frac{\theta}{\sigma^2}v^2\right) v^2 + C_2, & \text{if } |v| < v_*, \\ -\gamma v^2, & \text{if } |v| \geq v_*. \end{cases} \tag{49}$$

Next, we find v_*. By taking the gradient of $H(v)$, we get

$$H'(v) = -\frac{v}{\theta} + \left(\frac{\sigma}{\theta^{\frac{3}{2}}} - \frac{2\beta}{\sigma\sqrt{\theta}}\right) F\left(\frac{\sqrt{\theta}}{\sigma}v\right), \quad v \in (-v_*, v_*), \tag{50}$$

where

$$F(x) = e^{x^2} \int_0^x e^{-t^2} dt. \tag{51}$$

The boundary condition (46) implies that v_* is the root of

$$-\frac{v}{\theta} + \left(\frac{\sigma}{\theta^{\frac{3}{2}}} - \frac{2\beta}{\sigma\sqrt{\theta}}\right) F\left(\frac{\sqrt{\theta}}{\sigma}v\right) = -2\gamma v. \tag{52}$$

Substituting (13), (14), and (25) into (52), yields that v_* is the root of

$$(\mathsf{mse}_\infty - \beta)\, G\left(\frac{\sqrt{\theta}}{\sigma}v\right) = \mathsf{mse}_\infty - \mathsf{mse}_{Y_i}, \tag{53}$$

where $G(\cdot)$ is defined in (15). Solving (53), we obtain that v_* can be expressed as a function $v(\beta)$ of β, where $v(\beta)$ is defined by (18). By this, we obtain a candidate solution to (41).

2) Verification of the optimality of the candidate solution: Next, we use Itô's formula to verify the above candidate solution is indeed optimal, as stated in the following theorem:

Theorem 4. *For all $v \in \mathbb{R}$, $H(v)$ in* (49) *is the value function of the optimal stopping problem* (41). *The optimal stopping time for solving* (41) *is τ_* in* (43), *where $v_* = v(\beta)$ is given by* (18).

In order to prove Theorem 4, we have established the following properties of $H(v)$:

Lemma 5. $H(v) = \mathbb{E}_v\left[-\gamma V_{\tau_*}^2 - \int_0^{\tau_*}(V_s^2 - \beta)ds\right]$.

Lemma 6. $H(v) \geq -\gamma v^2$ *for all* $v \in \mathbb{R}$.

A function $f(v)$ is said to be *excessive* for the process V_t if

$$\mathbb{E}_v f(V_t) \leq f(v), \forall t \geq 0, v \in \mathbb{R}. \tag{54}$$

By using Itô's formula in stochastic calculus, we can obtain

Lemma 7. *The function $H(v)$ is excessive for the process V_t.*

The proof of Lemmas 5-7 are given in [49]. Now, we are ready to prove Theorem 4.

Proof of Theorem 4. In Lemmas 5-7, we have shown that $H(v) = \mathbb{E}_v\left[-\gamma V_{\tau_*}^2 - \int_0^{\tau_*}(V_s^2 - \beta)ds\right]$, $H(v) \geq -\gamma v^2$, and $H(v)$ is an excessive function. Moreover, from the proof of Lemma 5, we know that $\mathbb{E}_v[\tau_*] < \infty$ holds for all $v \in \mathbb{R}$. Hence, $\mathbb{P}_v(\tau_* < \infty) = 1$ for all $v \in \mathbb{R}$. These conditions and Theorem 1.11 in [13, Section 1.2] imply that τ_* is an optimal stopping time of (41). This completes the proof. □

V. Numerical Results

In this section, we evaluate the estimation error achieved by the following four sampling policies:

1. *Uniform sampling*: Periodic sampling with a period given by $S_{i+1} - S_i = 1/f_{\max}$.
2. *Zero-wait sampling* [1], [20]: The sampling policy given by

$$S_{i+1} = S_i + Y_i, \tag{55}$$

which is infeasible when $f_{\max} < 1/\mathbb{E}[Y_i]$.

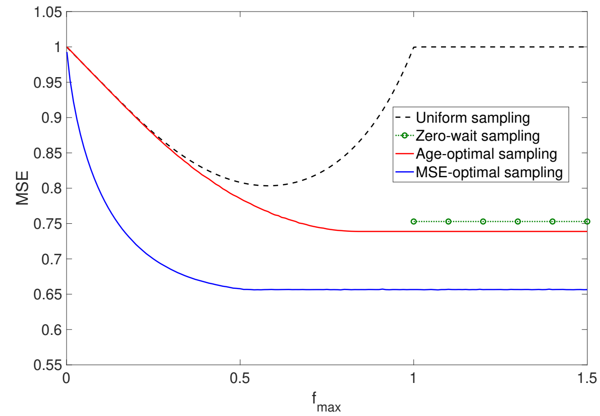

Fig. 3: MSE vs $f_{\max}$ tradeoff for *i.i.d.* exponential service time with $\mathbb{E}[Y_i] = 1$, where the parameters of the OU process are $\sigma = 1$ and $\theta = 0.5$.

3. *Age-optimal sampling* [15]: The sampling policy given by Theorem 3.
4. *MSE-optimal sampling*: The sampling policy given by Theorem 1.

Let $\mathsf{mse}_{\text{uniform}}$, $\mathsf{mse}_{\text{zero-wait}}$, $\mathsf{mse}_{\text{age-opt}}$, and $\mathsf{mse}_{\text{opt}}$, be the MSEs of uniform sampling, zero-wait sampling, age-optimal sampling, MSE-optimal sampling, respectively. We can obtain

$$\begin{aligned} &\mathsf{mse}_{Y_i} \leq \mathsf{mse}_{\text{opt}} \leq \mathsf{mse}_{\text{age-opt}} \leq \mathsf{mse}_{\text{uniform}} \leq \mathsf{mse}_\infty, \\ &\mathsf{mse}_{\text{age-opt}} \leq \mathsf{mse}_{\text{zero-wait}} \leq \mathsf{mse}_\infty, \end{aligned} \tag{56}$$

whenever zero-wait sampling is feasible, which fit with our numerical results.

Figure 3 illustrates the tradeoff between the MSE and $f_{\max}$ for *i.i.d.* exponential service times with mean $\mathbb{E}[Y_i] = 1$. Because $\mathbb{E}[Y_i] = 1$, the maximum throughput of the queue is 1. The parameters of the OU process are $\sigma = 1$, $\theta = 0.5$ and μ can be chosen arbitrarily. The lower bound mse_{Y_i} is 0.5 and the upper bound mse_∞ is 1. In fact, as Y_i is an exponential random variable with mean 1, $\frac{\sigma^2}{2\theta}(1 - e^{-2\theta Y_i})$ has a uniform distribution on $[0, 1]$. Hence, it is natural that $\mathsf{mse}_{Y_i} = 0.5$. For small values of $f_{\max}$, age-optimal sampling has similarity with uniform sampling, and hence $\mathsf{mse}_{\text{age-opt}}$ and $\mathsf{mse}_{\text{uniform}}$ are of same values. However, as $f_{\max}$ grows $\mathsf{mse}_{\text{uniform}}$ reaches the upper bound and remains constant for $f_{\max} \geq 1$. This is because the queue length of uniform sampling is large at high sampling frequencies. In particular, when $f_{\max} \geq 1$, the queue length of uniform sampling is infinite. On the other hand, $\mathsf{mse}_{\text{age-opt}}$ and $\mathsf{mse}_{\text{opt}}$ decrease with respect to $f_{\max}$. The reason behind this is that the set of feasible sampling policies satisfying the constraint in (10) and (35) becomes larger as $f_{\max}$ grows, and hence the optimal values of (10) and (35) are decreasing in $f_{\max}$. As we expected, $\mathsf{mse}_{\text{zero-wait}}$ is larger than $\mathsf{mse}_{\text{opt}}$ and $\mathsf{mse}_{\text{age-opt}}$. Moreover, all of them are between the lower bound and upper bound.

VI. Conclusion

In this paper, we have studied the optimal sampler design for remote estimation of OU processes through queues. We have developed optimal sampling policies that minimize the estimation error of OU processes subject to a sampling rate constraint. These optimal sampling policies have nice structures and are easy to compute. A connection between remote estimation and nonlinear age metrics has been found. The structural properties of the optimal sampling policies shed lights on the possible structure of the optimal sampler designs for more general signal models, such as Feller processes, which is an important future research direction.

References

[1] S. Kaul, R. D. Yates, and M. Gruteser, "Real-time status: How often should one update?" in *IEEE INFOCOM*, 2012.

[2] X. Song and J. W. S. Liu, "Performance of multiversion concurrency control algorithms in maintaining temporal consistency," in *Proceedings., Fourteenth Annual International Computer Software and Applications Conference*, Oct 1990, pp. 132–139.

[3] E. Altman, R. E. Azouzi, D. S. Menasché, and Y. Xu, "Forever young: Aging control for smartphones in hybrid networks," 2010, coRR, abs/1009.4733.

[4] Y. Sun, Y. Polyanskiy, and E. Uysal-Biyikoglu, "Remote estimation of the Wiener process over a channel with random delay," in *IEEE ISIT*, 2017.

[5] ——, "Sampling of the Wiener process for remote estimation over a channel with random delay," 2017, coRR, abs/1707.02531.

[6] G. E. Uhlenbeck and L. S. Ornstein, "On the theory of the Brownian motion," *Phys. Rev.*, vol. 36, pp. 823–841, Sept. 1930.

[7] J. L. Doob, "The Brownian movement and stochastic equations," *Annals of Mathematics*, vol. 43, no. 2, pp. 351–369, 1942.

[8] L. Evans, S. Keef, and J. Okunev, "Modelling real interest rates," *Journal of Banking and Finance*, vol. 18, no. 1, pp. 153 – 165, 1994.

[9] A. Cika, M.-A. Badiu, and J. P. Coon, "Quantifying link stability in ad hoc wireless networks subject to Ornstein-Uhlenbeck mobility," 2018, coRR, abs/1811.03410.

[10] H. Kim, J. Park, M. Bennis, and S. Kim, "Massive UAV-to-ground communication and its stable movement control: A mean-field approach," in *IEEE SPAWC*, June 2018, pp. 1–5.

[11] G. M. Lipsa and N. C. Martins, "Remote state estimation with communication costs for first-order LTI systems," *IEEE Trans. Auto. Control*, vol. 56, no. 9, pp. 2013–2025, Sept. 2011.

[12] E. Vinogradov, H. Sallouha, S. D. Bast, M. M. Azari, and S. Pollin, "Tutorial on UAV: A blue sky view on wireless communication," 2019, coRR, abs/1901.02306.

[13] G. Peskir and A. N. Shiryaev, *Optimal Stopping and Free-Boundary Problems*. Basel, Switzerland: Birkhäuswer Verlag, 2006.

[14] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, 5th ed. Springer-Verlag Berlin Heidelberg, 2000.

[15] Y. Sun and B. Cyr, "Sampling for data freshness optimization: Nonlinear age functions," accepted by *Journal of Communications and Networks*, 2019.

[16] ——, "Information aging through queues: A mutual information perspective," in *IEEE SPAWC Workshop*, 2018.

[17] Y. Xiao and Y. Sun, "A dynamic jamming game for real-time status updates," in *IEEE INFOCOM AoI Workshop*, April 2018, pp. 354–360.

[18] A. M. Bedewy, Y. Sun, S. Kompella, and N. B. Shroff, "Age-optimal sampling and transmission scheduling in multi-source systems," ACM MobiHoc, 2019.

[19] C. Kam, S. Kompella, G. D. Nguyen, and A. Ephremides, "Effect of message transmission path diversity on status age," *IEEE Trans. Inf. Theory*, vol. 62, no. 3, pp. 1360–1374, Mar. 2016.

[20] Y. Sun, E. Uysal-Biyikoglu, R. D. Yates, C. E. Koksal, and N. B. Shroff, "Update or wait: How to keep your data fresh," *IEEE Trans. Inf. Theory*, vol. 63, no. 11, pp. 7492–7508, Nov. 2017.

[21] C. Kam, S. Kompella, G. D. Nguyen, J. E. Wieselthier, and A. Ephremides, "On the age of information with packet deadlines," *IEEE Trans. Inf. Theory*, vol. 64, no. 9, pp. 6419–6428, Sept. 2018.

[22] R. D. Yates and S. K. Kaul, "The age of information: Real-time status updating by multiple sources," *IEEE Trans. Inf. Theory*, vol. 65, no. 3, pp. 1807–1827, Mar. 2019.

[23] Q. He, D. Yuan, and A. Ephremides, "Optimal link scheduling for age minimization in wireless systems," *IEEE Trans. Inf. Theory*, vol. 64, no. 7, pp. 5381–5394, July 2018.

[24] C. Joo and A. Eryilmaz, "Wireless scheduling for information freshness and synchrony: Drift-based design and heavy-traffic analysis," *IEEE/ACM Trans. Netw.*, vol. 26, no. 6, pp. 2556–2568, Dec 2018.

[25] A. M. Bedewy, Y. Sun, and N. B. Shroff, "Optimizing data freshness, throughput, and delay in multi-server information-update systems," in *IEEE ISIT*, 2016.

[26] ——, "Age-optimal information updates in multihop networks," in *IEEE ISIT*, 2017.

[27] ——, "Minimizing the age of the information through queues," accepted by *IEEE Trans. Inf. Theory*, 2019.

[28] ——, "The age of information in multihop networks," 2017, coRR, abs/1712.10061.

[29] Y. Sun, E. Uysal-Biyikoglu, and S. Kompella, "Age-optimal updates of multiple information flows," in *IEEE INFOCOM AoI Workshop*, 2018.

[30] I. Kadota, A. Sinha, and E. Modiano, "Optimizing age of information in wireless networks with throughput constraints," in *IEEE INFOCOM*, April 2018, pp. 1844–1852.

[31] R. Talak, S. Karaman, and E. Modiano, "Optimizing information freshness in wireless networks under general interference constraints," in *ACM MobiHoc*, 2018.

[32] N. Lu, B. Ji, and B. Li, "Age-based scheduling: Improving data freshness for wireless real-time traffic," in *ACM MobiHoc*, 2018.

[33] B. Zhou and W. Saad, "Joint status sampling and updating for minimizing age of information in the internet of things," 2018, coRR, abs/1807.04356.

[34] ——, "Minimizing age of information in the internet of things with non-uniform status packet sizes," 2018, coRR, abs/1901.07069.

[35] B. Hajek, K. Mitzel, and S. Yang, "Paging and registration in cellular networks: Jointly optimal policies and an iterative algorithm," *IEEE Trans. Inf. Theory*, vol. 54, no. 2, pp. 608–622, Feb 2008.

[36] M. Rabi, G. V. Moustakides, and J. S. Baras, "Adaptive sampling for linear state estimation," *SIAM Journal on Control and Optimization*, vol. 50, no. 2, pp. 672–702, 2012.

[37] A. Nayyar, T. Başar, D. Teneketzis, and V. V. Veeravalli, "Optimal strategies for communication and remote estimation with an energy harvesting sensor," *IEEE Trans. Auto. Control*, vol. 58, no. 9, pp. 2246–2260, Sept. 2013.

[38] K. Nar and T. Başar, "Sampling multidimensional Wiener processes," in *IEEE CDC*, Dec. 2014, pp. 3426–3431.

[39] X. Gao, E. Akyol, and T. Başar, "Optimal communication scheduling and remote estimation over an additive noise channel," *Automatica*, vol. 88, pp. 57 – 69, 2018.

[40] J. Chakravorty and A. Mahajan, "Remote estimation over a packet-drop channel with Markovian state," 2018, coRR, abs/1807.09706.

[41] P. J. Haas, *Stochastic Petri Nets: Modelling, Stability, Simulation*. New York, NY: Springer New York, 2002.

[42] S. M. Ross, *Applied Probability Models with Optimization Applications*. San Francisco, CA: Holden-Day, 1970.

[43] H. Mine and S. Osaki, *Markovian Decision Processes*. New York: Elsevier, 1970.

[44] D. Hayman and M. Sobel, *Stochastic models in Operations Research, Volume II: Stochastic Optimizations*. New York: McGraw-Hill, 1984.

[45] E. A. Feinberg, "Constrained semi-Markov decision processes with average rewards," *Zeitschrift für Operations Research*, vol. 39, no. 3, pp. 257–288, 1994.

[46] D. P. Bertsekas, *Dynamic Programming and Optimal Control*, 3rd ed. Belmont, MA: Athena Scientific, 2005, vol. 1.

[47] R. A. Maller, G. Müller, and A. Szimayer, "Ornstein-Uhlenbeck processes and extensions," in *Handbook of Financial Time Series*, T. Mikosch, J.-P. Kreiß, R. A. Davis, and T. G. Andersen, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 421–437.

[48] T. Soleymani, S. Hirche, and J. S. Baras, "Optimal information control in cyber-physical systems," *IFAC-PapersOnLine*, vol. 49, no. 22, pp. 1 – 6, 2016.

[49] T. Z. Ornee and Y. Sun, "Sampling for remote estimation through queues: Age of information and beyond," Technical Report, 2019, coRR, abs/1902.03552.

[50] H. Exton, *Handbook of hypergeometric integrals: Theory, applications, tables, computer programs*. Halsted Press, a division of J. Wiley, 1978.

[51] W. Dinkelbach, "On nonlinear fractional programming," *Management Science*, vol. 13, no. 7, pp. 492–498, 1967.

[52] D. P. Bertsekas, A. Nedić, and A. E. Ozdaglar, *Convex Analysis and Optimization*. Belmont, MA: Athena Scientific, 2003.

[53] H. Amann, *Ordinary Differential Equations An Introduction to Nonlinear Analysis*. Berlin: Walter De Gruyter, 1990.