Degrees of Freedom Per Communication Node

M. A. (Amir) Khojastepour*, K. Sundaresan*, M. Farajzadeh-Tehrani†, S. Rangarajan*
*NEC Laboratories America, Princeton, NJ, †Simons Center, Stony Brook University

Abstract

The classical definition of degrees of freedom (DoF) deals with the degrees of a communication channel or multiple communication channels in the limit of high SNR. This can be interpreted as the number of independent streams that can be sent in each communication channel in the high SNR regime. We introduce the concept of DoF per communication node where at a transmitting node the DoF is the number of independent dimensions that can be used for transmission and at each receiver node the DoF is the number of independent dimensions that can be used for receiving data signals. In general the communication channels or links in a network can be divided into two sets: the interfering channels and the intended channels; hence, the network may be considered as an overlay of two networks, respectively: the interfering network and data-intended network. In the classical form, DoF is defined for channels in the data-intended network. We illustrate a new interpretation of DoF that depends only on the interfering network and can be formalized in full generality based on degrees of freedom *per node* in the network. While the classical DoF has been studied generally in the context of interference and X-channels, the per node DoF concept generalizes the idea to other possible networks. Using this generalized notion of DoF, this paper provides new results on DoF for different networks and also makes a connection to the classical definition of DoF defined in interference and X-channels.

1. Introduction

In a multi-user wireless communication network, an intended signal transmission causes interference at the other receivers who are not involved in the transmission. While in a wired communication system these unwanted signals may be avoided, the shared nature of wireless medium makes interference one of the major limiting factors of its capacity when multiple (same band) transmissions are occurring simultaneously. Interference may be mitigated in different ways, including interference avoidance or cooperation between the communication nodes. When cooperation between the nodes is not possible or is very limited, interference alignment (IA) has proved to be a very effective technique [1]. The idea is to limit the interference from different sources to coincide in the same space whenever it cannot be totally avoided. Therefore the main purpose of IA is to minimize the interference spaces used over the whole network and hence maximize the size of intended signal dimension. Hence, the concept of degrees of freedom (DoF) in a multiuser channel provides a measure on the size of useful or intended signal dimensions in the network.

Interference alignment may be performed in different dimensions, e.g., in signal dimension [2] by using encoding techniques like lattice codes or in signal vector dimensions such as using coordinated precoding in MIMO systems[1]. The transmission of the signal might be considered in multiple sub-carriers or in multiple transmission blocks with independent fading, which allows the dimension of the signal vector to grow, allowing interference alignment to be performed more efficiently [1]. The IA based on symbol extension using time or frequency requires knowledge of all the extended channels before designing the precoders and receive filters. A more constrained but practically more appealing IA technique deals with a constant MIMO channel and formulates the problem in terms of finding fixed precoders and receiver filters that can minimize the interference[3]. Recent works have made significant progress on characterizing the DoF in interference channels with constant channel coefficients[4, 5, 6].

In this paper, we consider a new view of interference alignment in which we formulate DoF based on the transmission spaces at all transmitting nodes and interference spaces in all receiving nodes. Hence, we define DoF *per communication node* instead of the way it is classically defined with respect to communication links. Such a view allows us to decouple the interference network from the desired communication network, thereby helping solve for the DoF region in full generality for the interference network. An important advantage of this approach is that it allows us to compute

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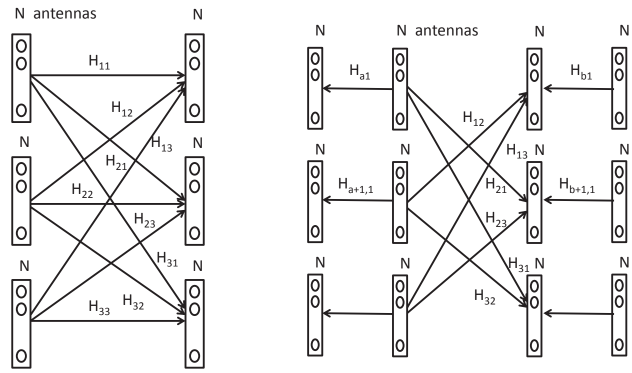
the achievable DoF for varied communication network topologies such as those whose links have asymmetric DoF and multi-user communication topologies involving multiple access and broadcast channels. Note that these cannot be easily accomplished under the conventional notion of DoF per link. By leveraging the notion of DoF per node, we also provide a network decomposition technique that helps reduce the interference network in stages, thereby helping compute the DoF as well as construct the IA solution for the interference network in an easy and simple manner. We apply our approach to several communication topologies and present our results on the DoF achievable in these various topologies, while also drawing a connection to the conventional model considered in interference channels.

The rest of the paper is organized as follows. With the help of current DoF per link model, we first motivate the benefit of defining DoF per node in section 2. Then, we formally define our notion of DoF per node in section 3, followed by our network decomposition technique as well as construction for IA solution in section 4. We then apply our approach to various communication network topologies in sections 5, 6, 7 and present our DoF results for the same. Finally, we present concluding remarks in section 8.

2. Motivation

DoF per Link: Consider a point to point channel between a transmitter and a receiver both equipped with multiple antennas. It is well known that for the independent Gaussian channel model between each pair of transmit and receive antennas, the capacity of the corresponding multiple antenna input and multiple antenna output (MIMO) channel scales with the minimum of the number of antennas at the transmitter (N_T) and the receiver (N_R) in the limit of high SNR[7]. The degrees of freedom of the channel is then defined as the quantity $\min(N_T, N_R)$. The concept of degree of freedom may also be interpreted as the possibility or measure of the number of independent streams that can be successfully transmitted simultaneously in the channel. It is immediate to see the usefulness of extending this concept to multiuser networks, where we are interested in understanding the number of simultaneous streams that can be transmitted between different subsets of transmit and receiver nodes in the network. For example, degrees of freedom in a three user interference channel with N antennas at each node is defined similarly as the scaling of the channel capacity between each pair of the users as a function of $\log(SNR)$. Specifically, it can be defined as the three tuple $\underline{d} = (d_1, d_2, d_3)$ that



(a) (b)

Figure 1. (a) 3-user MIMO interference channel. (b) Example of a modified desired network with the same interference network as in (a).

can be achieved simultaneously, where d_i denotes the scaling of the channel capacity between the i^{th} transmitter and receiver pair. While there may be multiple choices of $\underline{d}$ *achievable* in this network the region of all such $\underline{d}$ defines the *available* degrees of freedom region. Please note that for the rest of the paper we only consider constant channels for duration of the transmission which means that interference alignment may be performed only in vector signal space without channel extension.

Although the definition of DoF in general is a function of the actual channel gains, it is almost universally treated under generic or randomly generated channel conditions. As we discussed earlier, we do not consider symbol or channel extension, hence, we only consider space domain treatment of the signal where the channel coefficients are fixed. In practical scenarios, it means that we consider a precoder at the transmitter and a filter at the receiver per block or multiple blocks of transmission within the channel coherence time, where the channel coefficients are approximately constant. Consider an interference channel (Figure 1(a)) with 3 transmitting nodes indexed by 1, 2, and 3 and the corresponding receiving nodes denoted by 4, 5, and 6, respectively. The degrees of freedom corresponds to the rank of semi-orthogonal precoding matrices $\mathbf{V}_i$ and receive filters $\mathbf{U}_j$ such that the following condition holds [3]

$$\mathbf{U}_j \mathbf{H}_{ji} \mathbf{V}_i = 0 \quad \forall (i-j) \neq 0 (mod 3) \qquad (1)$$

$$\text{rank}(\mathbf{U}_j \mathbf{H}_{ji} \mathbf{V}_i) = d_i \quad \forall (i-j) = 0 (mod 3) \qquad (2)$$

where $\mathbf{U}_j$ is a $d_j \times N_j$ matrix, $\mathbf{V}_i$ is a $N_i \times d_i$ matrix and N_i is the number of antennas at the node i. It is not hard to see that if the channel matrices $\mathbf{H}_{ji}$ are generic satisfying the first set of conditions (1) is enough and the second set of conditions (2) are satisfied automatically. The classical approach to solve this problem assumes that the DoF per link $i, i = 1,2,3$ is d_i and the matrices $\mathbf{V}_i$ and U_i are of size $d_j \times N_j$ and $N_i \times d_i$, respectively.

Decoupling Interference and Data-intended Networks: The above example reveals an important observation that the DoF in such a network is just a function of the *interference network* (See Figure 2(a)) which is defined as a subset of the original network in which only the interfering links are present. In other words, the *desired* or *data-intended* network (See Figure 2(b)) that consists of the channels over which the actual communication and signal transmission takes place ($\mathbf{H}_{i+3,i}$ in the above example) does not play a direct role in the calculation of the DoF region in the network besides the fact that they enforce the condition on the size of matrices $\mathbf{V}_i$ and $\mathbf{U}_j$.

Now, let us turn to a modified network where the interference network remains the same but the data intended (or desired) network is replaced with another network consisting of six links with the component channels $\mathbf{H}_{i+6,i}, i = 1,\ldots,6$ as depicted in Figure 1(b) and further assume that the number of antennas of the new nodes satisfy $N_{i+6} \geq N_i$. Consider the question of finding the DoF in this network. Obviously, DoF in this network is characterized with 6 parameters, say d'_i one for each of the link $\mathbf{H}_{i+6,i}, i = 1,\ldots,6$, respectively. Is the solution obtained for the previous example, i.e. 3 user interference channel, applicable here? Can we say that DOF $d'_i, i = 1\ldots,3$ for the links $\mathbf{H}_{i+6,i}, i = 1,\ldots,3$ is equal to DoF $d_{i+3}, i = 1\ldots,3$ of the links $\mathbf{H}_{i+9,i+3}, i = 1,\ldots,3$ and is equal to $d_i, i = 1\ldots,3$, respectively? We will see later in Section 5, that it is indeed not the case, and present an example to show that d'_i or d'_{i+3} could be larger than d_i. Indeed, different data-intended networks can lead to different DoF. However, the key question we want to answer in this work is that, *given the striking similarity between the two problems (interference networks being the same) is there a solution that can encompass both scenarios?*

Idea and Approach: Our key idea and approach can be summarized as follows.

(*i*) We observe that Eqns. (1) and (2) suggest to decouple the problem of finding the DoF region in a network by considering a network as an overlay of two networks defined by the 'interference network' and the 'data intended (or desired) network'.

(*ii*) By defining the *DoF per node* in the network (and without any connection to the desired network topology), we solve the DoF region in the interference network in its full generality.

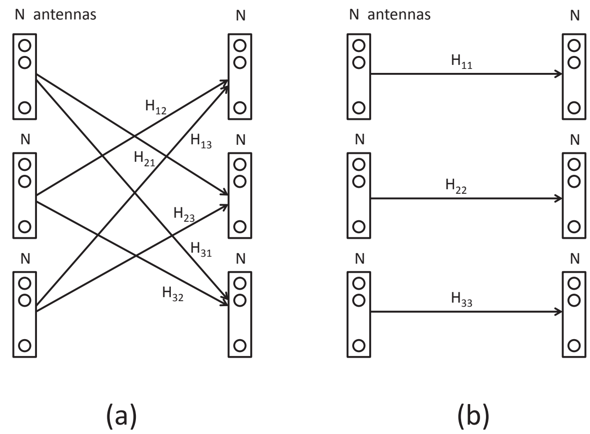


Figure 2. (a) The interference network for Figure 1(a). (b) The desired (or data intended) network for Figure 1(a)

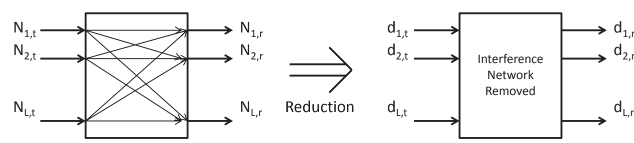


Figure 3. Illustration of reduction in the network.

(*iii*) We then abstract (replace) the interference network by a network with potentially lesser number of antennas defined by the DoF per node in the interference network, and ignore the interference network (i.e. make it a null network).

(*iv*) Finally, we consider the desired network and modify its channel coefficients based on the precoders and receive filters obtained as part of the second step.

By using a procedure that applies the above ideas one by one, in the end, we are left with a network without any interference but with possibly lesser dimensions in terms of the number of antennas and the size of channel matrices.

Network Decomposition: The concept of *DoF per node* allows us to use a decomposition technique that makes the problem of finding DoF in an interference network independent of the topology of the desired network. Furthermore, it allows for a *reduction* of the network (illustrated in Figure 3) by eliminating some of its edges as discussed above and making the problem simpler to address. We note that in reducing a network the interference edges are removed and the nodes are replaced with some *virtual antennas* where the number of virtual antennas at each node is equal to its DoF per node.

Applicability to Varied Topologies: We note that the classical definition of *DoF per link* in the network is still a function of the communication network as well. For example in the same 3-user interference channel we can consider a desired network that is defined by adding two new nodes 7 and 8. The desired network is then defined as a combination of two channels; a multiple access channel from the transmitting nodes 1, 2, and 3 to a receiving node 7 denoted by the component channels $\mathbf{H}_{7i}, i = 1,2,3$ and a broadcast channel from a single point 8 to the receiving nodes 4, 5, and 6 denoted by the component channels $\mathbf{H}_{i8}, i = 4,5,6$. Clearly in this network the total DoF is a function of the number of antennas at the nodes 7 and 8 as well. Nonetheless, the treatment of the problem as two overlay networks of 'interference' and 'desired' network allows us to decouple the problem and also interpret the solution more easily. Also, it is possible to consider more general cases of the data intended network such as multiple access channel (MAC) and broadcast channel (BC) as illustrated in the above example which is beyond the classical definition of DoF that is generally considered in the context of classical interference channel and X-channels.

3. Degrees of Freedom per Node

In this section we formally define *DoF per node* in a communication network. Consider a network of L nodes equipped with N_i, $i = 1,2,\ldots,L$ antennas. Each node serves either as a transmitter or a receiver. The communication channel defined as an oriented graph of edges $\mathscr{E}$ on the set of nodes where the component channel between different nodes is assumed to be a Gaussian channel denoted by the channel coefficients matrix $\mathbf{H}_{ji}$ with complex entries from the transmitting node i to the receiving node j. A component channel does not exist in the graph if its channel matrix is zero. The receive signal at a receiving node j is defined as

$$\mathbf{y}_j = \sum_{i \in \mathscr{T}} \mathbf{H}_{ji}\mathbf{x}_i + \mathbf{z}_j, \forall\ j \in \mathscr{R} \tag{3}$$

where $\mathscr{T}$ is the set of transmitting node indices, $\mathscr{R}$ is the set of receiving node indices, $\mathbf{y}_j$ is the received signal at the receiving node j, $\mathbf{x}_i$ is the transmitting signal at the transmitting node i, and $\mathbf{z}_i$ is the Gaussian noise at the receiver of node j.

The set of component channels is divided into two sets: a set $\mathscr{D}$ consisting of the data intended (or desired) link and its complement set $\mathscr{I}$, ($\mathscr{I} \cup \mathscr{D} = \mathscr{E}$ and $\mathscr{I} \cap \mathscr{D} = \varnothing$), that consists of the links whose output only causes interference at the receiving node and their corresponding signal does not carry any intended data to this node.[1]

We say a vector of $\underline{d} = (d_1,\ldots,d_L)$ DoF per node for the nodes $1,\ldots,L$ is achievable if and only if there exists a set of transmit precoders $\mathbf{V}_i$ of size $N_i \times d_i$ for the nodes $i \in \mathscr{T}$ and a set of receive filters $\mathbf{U}_j$ of size $d_j \times N_j$ for the receiving nodes $j \in \mathscr{R}$ such that $\mathbf{U}_j\mathbf{H}_{ji}\mathbf{V}_i = 0$ simultaneously. Please note that by definition a precoder and a receive filter is a full rank semi-orthogonal matrix.

Alternative definition: The above definition of per node DoF is equivalent to the following: Let the interference network be defined by the graph $(\mathscr{T} \cup \mathscr{R}, \mathscr{I})$ with a total of L nodes which are indexed by $\mathscr{T}$ for the transmitting nodes and $\mathscr{R}$ for the receiving nodes. Let us amend this interference network with a desired network that consists of L links defined by the set $\mathscr{E}'$ and L extra nodes indexed by $\mathscr{T}' \cup \mathscr{R}'$ such that one link connects each transmitting node $i \in \mathscr{T}$ to a different node in the set of new nodes $\mathscr{T}'$ and one link connects each node from the set of new nodes $\mathscr{R}'$ to a different receiving node in $\mathscr{R}$. we further assume that the number of antennas of the new nodes are the same as that of the nodes they are connected to. We say a vector of $\underline{d} = (d_1,\ldots,d_L)$ DoF per node for the nodes $1,\ldots,L$ is achievable if and only if there exists a coding scheme, which achieves the capacity scaling of $d_i \log(SNR) + o(\log(SNR))$ in the limit of high SNR simultaneously for all the corresponding links in the set of new links $\mathscr{E}'$ in the desired network for the generic choice of all channels.

The above definition clarifies that once a vector of DOF per node $\underline{d}$ is achievable it is possible to remove the interference network from the original network and replace the number of antennas at each node to d_i instead of N_i (see Figure 3) and update the component channel coefficients by right and left multiplication with the corresponding precoder and receive filter of transmit and receive nodes of this component channel, respectively. This change would not affect the DoF in the reminder of the network that is defined to be the desired

[1] Later in Section 4 we discuss a case that the desired link and interference link are defined partially in which we only consider a partial set of interference links whose interference we would like to cancel in this stage and the set of desired link might still contain some interference link that we may consider at a later stage.

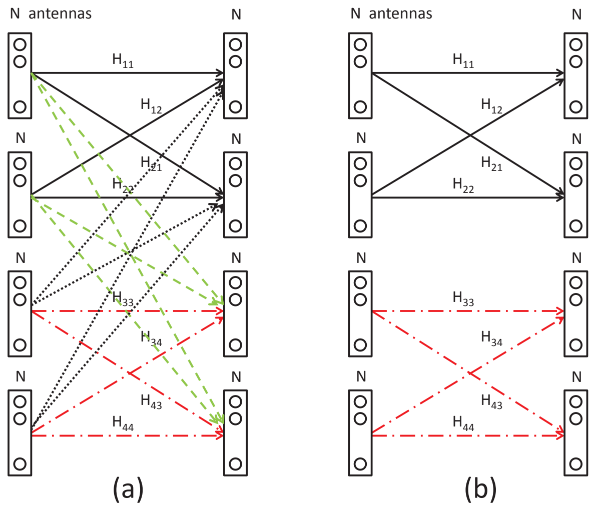


Figure 4. (a) FBIN(4,4). (b) Partial interference network of FBIN(4,4) as union of two FBIN(2,2).

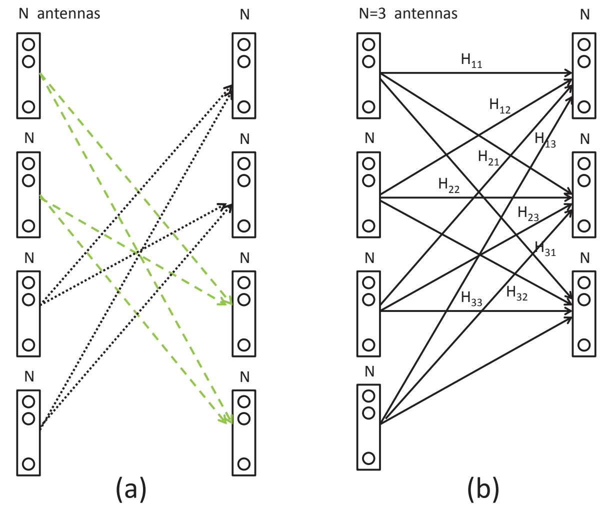


Figure 5. (a) a different partial interference network for FBIN(4,4). (b) FBIN(4,3).

network. We point out that our analysis is solely with respect to the DoF in the network and such reduction may affect the actual capacity region of the channel in a different way. In particular, even different solutions for the precoders and receiver filters that correspond to the same DoF per node may also affect the desired network in such a way that the achievable capacity or throughput in the desired network is different. Nonetheless, in terms of high SNR analysis, the reduction obtained by the notion of DoF per node and removal of the corresponding interference network does not change the capacity scaling.

4. Flexible Application of DoF Per Node

Flexible Construction: Often computing an interference alignment solution and hence determining the DoF supported by a large interference network is a very challenging problem. The concept of DoF per node helps tackle this problem in a flexible and modular way. We illustrate this with the help of an example. Consider the problem of calculating an achievable DoF per node in an interference network defined as FBIN(4,4), depicted in Figure 4(a), with $N = 4$ antennas at each node, where FBIN(L,K) denotes a full bipartite interference network from a set of L transmitting nodes to a set of K receiving nodes with LK component channels, the latter being between each pair of transmitting and receiving nodes. One may consider this network as the overlay of an interference network defined as a part of FBIN(4,4), consisting of two FBIN(2,2) network: one defined from the first two transmitting nodes to the first two receiving nodes, and the other defined from the last two transmitting nodes to the last two receiving nodes (as shown in Figure 4(b)), with a desired network defined by the complement of this network. We can easily deduce that DoF per node of 2 is achievable for all nodes. Hence, we can remove the interference network and consider the network with only two antennas at each node and update the component channels respectively. Please note that for the generic choice of channel conditions, the channels remain generic with this update process. After the update, we have a network that consists of exactly two FBIN(2,2): one defined from the first two transmitting nodes to the last two receiving nodes and one from the last two transmitting nodes to the first two receiving nodes (as shown in Figure 5(a)). Hence, it is now immediate that DoF per node of one is achievable for all the nodes in this network which means that the same DoF per node (i.e. 1) is achievable for the original FBIN(4,4). Thus, the flexible nature of DoF per node allows for an easier computation of DoF in a large interference network by sequentially reducing it to smaller interference networks.

Sub-optimality: Note that although it is convenient to partially or successively apply the tools of calculating DoF per node and reducing the problem dimension gradually, this may result in not identifying all the achievable DoF in the original network. We will illustrate two specific examples.

(1) In FBIN(4,4), a higher DoF per node may be achievable with $N = 4$. Specifically, it can be shown that DoF per node of 2 for two transmitting nodes and DoF per node of 1 for all other nodes are achievable for

FBIN(4,4) with $N = 4$ antennas per node.
(2) A more interesting result is that in FBIN(4,4) even with $N = 3$ antennas a DoF per node of 1 is achievable, which however cannot be computed from the approach outlined earlier (which requires $N = 4$ antennas). The construction for this is as follows. We consider the orientation for the edges of the network which goes from the first two transmitters with precoders $\mathbf{V}_1, \mathbf{V}_2$ to the first two receivers with receive filters $\mathbf{U}_1, \mathbf{U}_2$. Then, we consider a reciprocal channel that goes from the first two receiving nodes with the precoders $\mathbf{U}_1, \mathbf{U}_2$ (receive filters in the original direction) to the last two transmitting nodes with the receive filters $\mathbf{V}_3, \mathbf{V}_4$ (that are the precoders in the original direction). Next, we consider the channel from the last two transmitting nodes with precoders $\mathbf{V}_3, \mathbf{V}_4$ to the last two receiving nodes with the receive filters $\mathbf{U}_3, \mathbf{U}_4$, and finally we consider the reciprocal channel from the last two receiving nodes with precoders $\mathbf{U}_3, \mathbf{U}_4$ to the first two transmitting nodes with precoders $\mathbf{V}_1, \mathbf{V}_2$. Then,

(*i*) $\mathbf{U}_j = \mathbf{H}_{j1}\mathbf{V}_1 \times \mathbf{H}_{j2}\mathbf{V}_2$, if $1 \leq j \leq 2$;

(*ii*) $\mathbf{U}_j = \mathbf{H}_{j1}\mathbf{V}_3 \times \mathbf{H}_{j4}\mathbf{V}_4$, if $3 \leq j \leq 4$;

(*iii*) $\mathbf{V}_i = \mathbf{H}^*_{1i}\mathbf{U}_1 \times \mathbf{H}^*_{2i}\mathbf{U}_2$, if $3 \leq i \leq 4$;

(*iv*) $\mathbf{V}_i = \mathbf{H}^*_{3i}\mathbf{U}_3 \times \mathbf{H}^*_{4i}\mathbf{U}_4$, if $1 \leq i \leq 2$.

where '$\times$' is the standard curl operation between two vectors and H^* denotes the channel reciprocal to channel H. For example, the conditions (i) state that the space represented by $\mathbf{U}_j$ is a one dimensional space that is orthogonal to the two dimensional space defined by the vectors $\mathbf{H}_{j1}\mathbf{V}_1$ and $\mathbf{H}_{j2}\mathbf{V}_2$, etc.

Putting together, we get

(*i*) $\mathbf{V}_i = \mathbf{H}^*_{1i}(\mathbf{H}_{11}\mathbf{V}_1 \times \mathbf{H}_{12}\mathbf{V}_2) \times \mathbf{H}^*_{2i}(\mathbf{H}_{21}\mathbf{V}_1 \times \mathbf{H}_{22}\mathbf{V}_2)$, if $3 \leq i \leq 4$;

(*ii*) $\mathbf{V}_i = \mathbf{H}^*_{3i}(\mathbf{H}_{33}\mathbf{V}_3 \times \mathbf{H}_{34}\mathbf{V}_4) \times \mathbf{H}^*_{4i}(\mathbf{H}_{43}\mathbf{V}_3 \times \mathbf{H}_{44}\mathbf{V}_4)$, if $1 \leq i \leq 2$.

We can simplify this further and get an equation only involving $\mathbf{V}_1$ and $\mathbf{V}_2$ which can be directly solved. Then, we find $\mathbf{U}_1, \mathbf{U}_2$ from (i), $\mathbf{V}_3, \mathbf{V}_4$ from (iii), and finally $\mathbf{U}_3, \mathbf{U}_4$ from (ii). This is a direct construction of one dimensional precoders and receive filters that satisfy the interference alignment condition and shows that a per node DoF of 1 is achievable for all nodes.

5. Symmetric Interference Network with Asymmetric DOF

Network Topology: Let us consider a 3-user interference channel with $\mathscr{T} = \{1,2,3\}, \mathscr{R} = \{4,5,6\}$ where the node i intends to communicate with the node $i+3$, i.e., $\mathscr{D} = \{(1,4),(2,5),(3,6)\}$ and the number of antennas at all nodes is $N = 3$.

DoF Result: It is known that a DoF equal to 1 for all three links in $\mathscr{D}$ is achievable in the conventional interference channel. Here, we argue that one can achieve a total DoF of 4 over all three links under our DoF per node model.

Construction: Let us consider the partial interference network defined by $\mathscr{I}' = \{(1,5),(1,6),(2,6),(3,5)\} \subset \mathscr{I} = \mathscr{T} \times \mathscr{R} - \mathscr{D}$. We argue that the vector $\underline{d} = (2,2,2,3,1,1)$ DoF per node is achievable in this interference network. Consider an arbitrary precoder $\mathbf{V}_1$ of size 3×2 which defines a two dimensional space as an input to either of the channels from node 1 to nodes 5 and 6. Since $N = 3$ at each of the nodes 5 and 6, there is at least one vector (channels are generic) that is orthogonal to the received signal from node 1 based on which we define the receive filters $\mathbf{U}_5$ and $\mathbf{U}_6$ as a 1×3 dimensional vector. Now consider the input to node 2 that lies in a 3 dimensional space. This input should avoid generating an output at node 6 that corresponds to the vector defined by $\mathbf{U}_6$. So for generic choices of the channels there is a two dimensional space defined for example by the basis corresponding to the columns of $\mathbf{V}_2$ that does not produce any vector corresponding to $\mathbf{U}_6$ at node 6. Similar argument holds for node 3 by considering that the only interfering link out of this node goes to node 5.

Next, we can consider the rest of the interfering network by omitting the links $\mathscr{I}'$ and replacing the number of antennas at the nodes 1,2, ...6 by the corresponding DoF per nodes, i.e., 2, 2, 2, 3, 1, 1, respectively. We note that the interfering network in this case consists of only two links $\mathscr{I}'' = \mathscr{I} - \mathscr{I}' = \{(2,4),(3,4)\}$. Considering the fact that the modified number of antennas at the node 2,3, and 4, are equal to 2, 2, and 3, it is simple to see that DoF per node of 1, 1, and 2 is achievable. This completes the proof of showing that DoF per node of $(2,1,1,2,1,1)$ is achievable.

Considering the desired network defined by the edges in $\mathscr{D}$, it can be deduced that the DoF for each link is the minimum of per node DoF of the nodes at the two ends of this link. This means that for the first link, DoF is equal to 2 and for the other two links, their DoF is equal to 1, resulting in total of 4 DoF for all three communication links in this network.

Extensibility: Recall that one of the advantages of applying the DoF per node model is that is helps decouple the interference network from the data-intended network. Hence, the DoF per node achievable from our construction above, can be used to understand the actual

DoF achievable under various communication network configurations. Specifically, it is also possible to show a vector of $(2,2,2,1,1,1)$ DoF per node is achievable which is asymmetric in terms of the total transmit and receive degrees of freedom in the network. Hence, considering the desired network to be defined by the set of edges in $\mathscr{D}$ (as defined above) the DoF per communication link remains to be one for all the three links. However, considering a different desired network defined by $\mathscr{D}_1 = \{(1,7),(2,7),(3,7),(8,4),(8,5),(8,6)\}$ with node 8 as transmitter and node 7 as receiver overlayed on top of the same interference network $\mathscr{I}$,it is easy to see that total DoF for the entire communication links in this network is equal to 6 in the multiple access channel from nodes 1,2, and 3 to node 7 and it is equal to 3 for the broadcast channel from node 8 to nodes 4, 5, and 6 for a total of 9 DoF.

6. Asymmetric Interference network

Network Topology: Here, we consider an asymmetric interference network with three transmitters $\mathscr{T} = \{1,2,3\}$ and four receivers $\mathscr{R} = \{4,5,6,7\}$ and all the channels between every transmitter to the receiver. This channel is reciprocal to the channel depicted in Figure 5(b). We assume that all nodes have 3 antennas.

DoF Result: While computing achievable DoF in such asymmetric communication topologies cannot be easily accomplished under the conventional DoF per link model, we show that a DoF per node of 1 is achievable with the help of our model.

Construction: We will only focus on the interference network without a particular desired network and we would like to find an achievable DoF per node for this interference network. First let us define a partial interference network that consist of the edges between the first 6 nodes of the network and does not include the edges in $\mathscr{I}' = \{(1,7),(2,7),(3,7)\}$. It was seen in Section 5 that DoF per node of $(2,2,2,1,1,1)$ is achievable for the nodes 1 to 6 and since node 7 is an isolated node in this partial interference network without connection to any other node, it is immediate that the vector of achievable DoF per node in the partial interference network is $(2,2,2,1,1,1,3)$. Now, by reducing the interference network to a network with $(2,2,2,1,1,1,3)$ antennas for nodes 1 to 7, respectively, and with the edges defined by $\mathscr{I}'$, we can see that the problem reduces to finding DoF per node in a multiple access network with three transmitting nodes with two antennas each to a single destination with 3 antennas. It is very simple to see that DoF of 1 per communication node is possible. Hence, a DoF of 1 per node for the original network with three transmit and four receive antennas

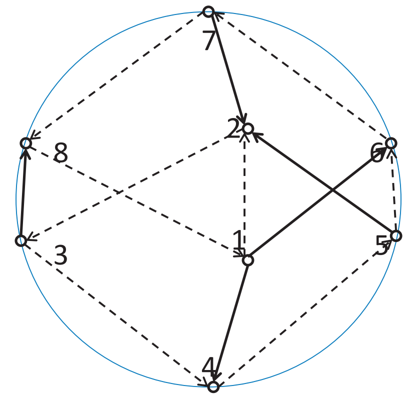


Figure 6. A physical scenario and positioning of communication nodes with desired network consisting of a MAC, a BC, and a single link channel.

with channels connecting every pair of transmit and receive nodes is achievable.

7. Interference Network Overlayed with MAC and BC

Network Topology: Consider a communication network depicted in Figure 6 consisting of 4 transmitting nodes labeled as 1, 3, 5, and 7 communicating with 4 receiving nodes labeled as 2, 4, 6, and 8 that are located in an area and on the border of a disk. The communication channel between a pair of transmitter and receiver is assumed to be independent fading (satisfying generic channel conditions) with the average channel gain based on the distance between the nodes. The nodes are considered to be out of interference range if the distance between the nodes is larger than, say, 3/4 of the diameter of the disk. Hence, for the network topology depicted in Figure 6, the nodes 4, 6, and 8 are out of range to receive interference from nodes 7, 3, and 5, respectively.

DoF Result: While computing achievable DoF in such multi-user communication topologies involving multiple access and broadcast channels simultaneously cannot be easily accomplished under the conventional DoF per link model, we show that a DoF per node of two is achievable with the help of our model.

Construction: We define the desired network as a combination of (i) a broadcast channel (BC) from node 1 to nodes 4 and 6, (ii) a multiple access channel (MAC) from the nodes 5 and 7 to node 2, and (iii) a single link channel from node 3 to node 8. The signal received from a transmitting link at all other links beside the ones in the desired network are considered to be interference and the corresponding links define the interference network. For our example these links are $(1,2),(3,2),(3,4),(5,4),(5,6),(7,6),(7,8)$, and $(1,8)$. Assume that each node has 4 antennas. The nodes can achieve a DoF 2 for nodes in this interference network. The solution can be obtained as follows. The precoders $\mathbf{V}_1,\mathbf{V}_3,\mathbf{V}_5,\mathbf{V}_7$ must satisfy the following equations where $\overset{S}{=}$ means that the vector spaces defined by the column of the matrices in the left and right of this operator have to be the same (have same span).

$$\mathbf{H}_{21}\mathbf{V}_1 \overset{S}{=} \mathbf{H}_{23}\mathbf{V}_3 \tag{4}$$

$$\mathbf{H}_{43}\mathbf{V}_3 \overset{S}{=} \mathbf{H}_{45}\mathbf{V}_5 \tag{5}$$

$$\mathbf{H}_{65}\mathbf{V}_5 \overset{S}{=} \mathbf{H}_{67}\mathbf{V}_7 \tag{6}$$

$$\mathbf{H}_{87}\mathbf{V}_7 \overset{S}{=} \mathbf{H}_{81}\mathbf{V}_1 \tag{7}$$

Hence, it is enough to have

$$\mathbf{V}_1 \overset{S}{=} \mathbf{H}_{21}^{-1}\mathbf{H}_{23}\mathbf{H}_{43}^{-1}\mathbf{H}_{45}\mathbf{H}_{65}^{-1}\mathbf{H}_{67}\mathbf{H}_{87}^{-1}\mathbf{H}_{81}\mathbf{V}_1 = \mathbf{H}_c\mathbf{V}_1 \tag{8}$$

which means that the columns of $\mathbf{V}_1$ should be the eigenvectors of the matrix $\mathbf{H}_c$. Hence, we can simply pick any two eigenvectors of the matrix $\mathbf{H}_c$ to form the precoding matrix $\mathbf{V}_1$. The precoders $\mathbf{V}_3$, $\mathbf{V}_5$, and $\mathbf{V}_7$ are then obtained successively based on the above equations. Finding the receive filters is also very easy once the transmit precoders are fixed. The above construction does not limit the number of eigenvectors that can be picked, however, the space of the receive filters would decrease as we increase the size of the space of the precoders. For example by picking 3 eigenvectors to form $\mathbf{V}_1$, the other precoder would also be equivalent to a 3-dimensional subspace which then limits each of the receive filter to lie in a 1-dimensional space which translates to the achievable per node DoF vector of $(3,1,3,1,3,1,3,1)$.

Having DoF of 2 per node for all nodes, we can now eliminate the interference network and focus on the desired network. For example an interesting observation here could be the fact that in the MAC channel part of this network, no more than two streams can be decoded, but it is possible to use precoders at the transmitting nodes to optimize the rate. We note that the design of the precoders depends on the updated channels from the nodes 5 and 7 to node 2 which is composed of two 2×2 channels.

8. Conclusion

In this work, we introduced a new interpretation of degrees of freedom (DoF) in the network. Classically DoF is defined for a collection of independent links in the network. In contrast, we introduced the notion of *DoF per node*. Furthermore, we introduced the technique of dividing a network into a desired network and an interference network, where the former consists of the links over which the actual data transmission is performed and the latter consists of the links that do not carry information and only cause interference to their corresponding receivers. We tied back our definition of DoF per node to the classical concept of DoF per link and further showed its usefulness in several new problems and topologies. This included simpler interpretation of DoF, decoupled handling of interference and communication in the network, successive application of interference removal by applying the technique over partial interference networks, and considering several new scenarios and topologies for the interference alignment problem.

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