A Stochastic Examination of the Interference in Heterogeneous Radio Access Networks

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***Abstract*—Networks of the future have been forecast as being increasingly heterogeneous. Heterogeneous networks, in general, display a high degree of spatial randomness which is a result of the deployment of a variety of smaller base stations (BSs) along with macro BSs wherever and whenever it is deemed necessary. An important aspect of analyzing, improving, and deploying heterogeneous networks involves understanding the interference that users in such a network encounter. This paper documents a part of our work towards such an analysis of the efficiency of heterogeneous networks. The work delineated in this paper discerns the interference experienced in a heterogeneous network consisting of macro- and micro- BSs which are considered to be points of a Poisson cluster process in the Euclidean plane. This paper has two main contributions: the first is an expression for the generating functional of the interference (akin to its density function) and the second is a proof that shows that the interference in any given area can be approximated to be completely described by a Gaussian distribution with zero mean and a variance that is dependent on the number of BSs in the area, along with the functionals that represent the pathloss, transmit power, fading, etc. which are ascribed to each BS.**

I. Introduction

A general consensus, as evinced by numerous papers from academia and the industry alike, has emerged wherein Heterogeneous Networks (HetNets) are considered to be the networks of the future. HetNets, in general, consist of different types of BSs (like macro, micro, pico, femto, etc.) which are deployed with a fairly irregular topology. The deployment is generally based on user densities and user demands in a given part of the area, i.e. micro BSs are deployed in locations where the macro BSs are unable to satisfy user demands. Initial efforts to understand HetNets involved running extensive simulations by fixing macro BSs on a hexagonal grid and randomly distributing micro BSs within the cell areas of the macro BSs. A completely theoretic examination has been a fairly recent endeavor.

Mathematical analysis of such scenarios is a very complicated task, and simplified system level models have been used to understand the relationship between different parameters of interest such as probability of coverage (or outage), or spectral efficiency, etc. with the overall efficiency or optimality of the network. One of the many methods used for the analysis of a system is stochastic geometry where BSs and users are considered to be points of a point process. Various functionals describing network (or BS) attributes such as fading, path loss, transmit power, etc. are ascribed to each point of these point processes. This method of mathematical scrutiny has been used to achieve some very insightful results for homogeneous networks as elucidated in [1]–[3]. H.S.Dhillon, et.al., in [4], extend this approach of analysis of homogeneous networks to HetNets by considering HetNets as networks consisting of multiple layers of homogeneous networks to underscore some very important insights regarding the probability of coverage in such networks.

Our approach considers the HetNet to be a clustered point process where the micro BSs are clustered around the macro BS. The way the micro BSs are clustered is determined by a simple distribution such as a uniform distribution or Gaussian distribution which, in turn, reflects the distribution of users within a particular sub-area. The distribution of the micro BSs around the macro BSs is driven by (or dependent on) the user density in the area. Simply put, the intensity of micro BSs can be assumed to be a function of the user intensity. Throughout this work, we consider the distribution of the micro BSs around the macro BS to be as generic as possible and it is frequently denoted by $f(\cdot)$. The macro BSs (centers of the clusters) are treated as points of a Poisson Point Process (PPP). Such processes are generally known as Poisson Clustered Processes (PCPs). For the first main contribution of this paper, in Sections II-A and II-B, we consider the PCPs to be Neyman-Scott processes which result from homogeneous independent clustering applied to a stationary Poisson process. Therefore, as discussed in [5] and [6], the PCPs considered in this paper are special cases of doubly stochastic processes or Cox processes.

A Cox process is driven by a random field; in our examinations, the random field is considered to be stationary which results in a stationary clustered process. Further details about the consequences of this assumption are given in the beginning of Section II. The differences between the two processes mentioned here can be observed in Figures 1 and 2. The Figure 1 shows a realization of a homogeneous PPP which has been used in [1]–[4]. Note that all the points realized are of the same type, which implies that there is only one type of BS considered per layer. This work uses PCPs, a realization of which is illustrated in Figure 2, where two types of BSs can be considered per layer. A PCP allows clustering of one type of points around a point of another type. This implies, a more realistic representation of a real world network which can mimic the clustered deployment of micro BSs around a macro BS can be realized using this method. We use this framework to assess the efficiency of a HetNet based on a capacity constraint. A task that is essential for achieving this target is that of understanding the behavior of the interference within such a framework. The main results of this paper are dedicated to understanding the behavior of the interference experienced by a user in a network that is exemplified by this framework.

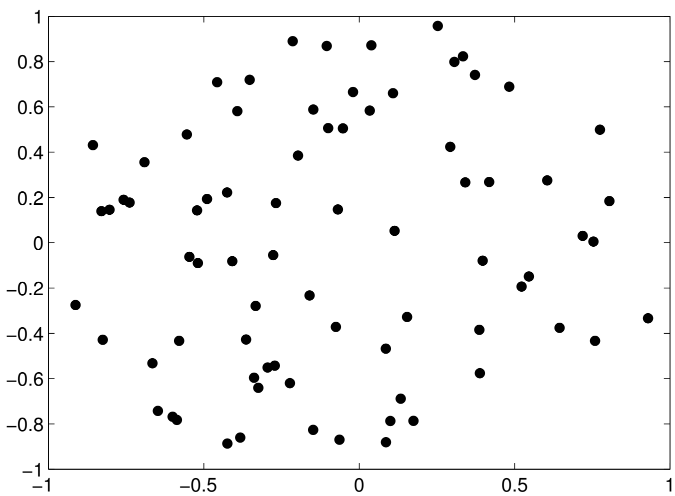

Fig. 1. Homogeneous PPP with intensity $\lambda = 25$

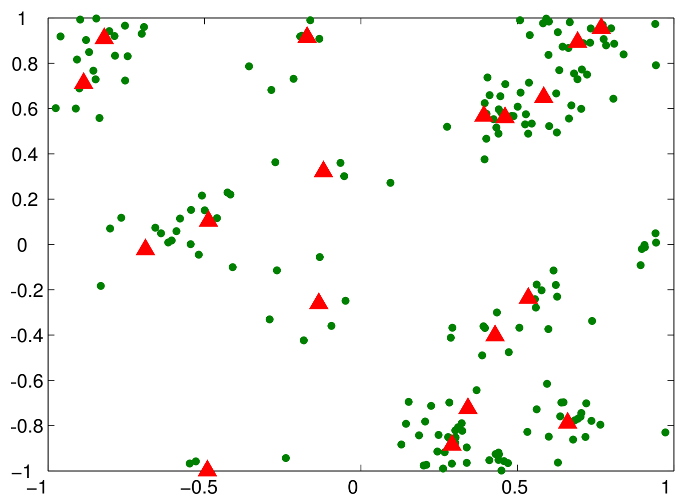

Fig. 2. PCP with $\lambda = 25$ and 10 cluster points per parent

This paper is organized as follows: The opening paragraphs of Section II contain the system model. The system model described in Section II (below) forms the basis for our initial goal of trying to estimate the energy consumption of a heterogeneous network based on a capacity constraint and led to the findings documented in this paper. Section II-A illustrates how the density function of the interference can be described. This is then used in Section II-B where the expression for the generating functional of the interference (or the density function of the interference) is derived and this is one of the two main findings of our paper. Section II-C describes the asymptotic behavior of the generating functional of the interference and contains a theorem which is the other salient find. Lastly, Section III contains the conclusion and future work of the authors.

II. Downlink System Model

The downlink system model considers macro- and micro-BSs as points of a stationary Poisson cluster process (a special case of a doubly stochastic or Cox process). The assumption of stationarity results in a fixed number of points per cluster and a fixed normalized kernel "bandwidth"[1]. This implies that, the average number of micro BSs per cluster, though scattered according to a particular distribution, are fixed. This assumption also implies that the spread of the cluster (or the maximum radius up to which micro BSs are present around a particular macro BS) is fixed. The *single antenna* macro BSs are distributed according to a homogeneous Poisson point process (PPP) Φ_c with density λ_c around which the *single antenna* micro BSs are clustered based on a given distribution $f(\cdot)$. Throughout this work, the cluster intensity is denoted by λ_m and is assumed to be fixed. This stems from the assumption of stationarity and implies that the average number of micro BSs distributed around a particular macro BS in any realization is fixed. For a point process Φ, the Voronoi cell determines the area covered by a particular BS and is given by

$$\begin{aligned}\mathcal{C}_{X_b} &= \left\{z \in \mathbb{R}^2 : \text{SINR}_z \geq T\right\} \\ &= \left\{z \in \mathbb{R}^2 : L\left(z, x_i\right) \geq T\left(I_\Phi\left(z\right) + W\right)\right\},\end{aligned}$$

where $X_b = \{x_i\} \in \mathbb{R}^2$ is the set of BS locations, T is a threshold, and SINR_z is the Signal-to-Interference-plus-Noise ratio at point $z \in \mathbb{R}^2$. $L(z, x_i)$ is power received at point z, W is the noise power, and $I_\Phi(z)$ is the interference at point z. The noise power is assumed to be additive and constant with σ^2, and no assumptions are made about its distribution. The interference and the received power are given by

$$I_\Phi\left(z\right) = \sum_{\substack{x_j \in \Phi \\ j \neq i}} L\left(z, x_j\right), \text{ and } L(z, x_i) = \frac{Ph}{l\left(|x_i - z|\right)},$$

respectively; where P is the transmit power, h is a fading parameter defined as an exponential random variable with a mean μ, and $l\left(|x_i - z|\right)$ is the path loss function usually considered to be in a power law form such as $||x_i - z||^\beta$, or $(1 + ||x_i - z||)^\beta$. Here, β is the path loss exponent and is considered to be greater than 2 (i.e. $\beta > 2$). This model assumes that the user at a given location $z \in \mathbb{R}^2$ connects to the BS (macro or micro BS) closest to it. It should be noted that, for mathematical simplicity, an identical pathloss

[1]Note: "bandwidth" is a term from mathematical literature (for ex: in [7] and references therein) which defines the spread of the cluster points around the parent point and shouldn't be confused with the definition of bandwidth in communications engineering.

model is assumed for macro and micro BSs. The transmit power, P, can be described by a two point distribution to account for different transmit powers of macro and micro BSs. However, the description of the transmit power P isn't central to the results obtained in this paper. It should be stressed that altering the assumptions (the identical pathloss model or a two point distribution for the transmit power) would not change the results documented in this paper in anyway. The spatially averaged rate for a point process with intensity λ, as seen in [2], [3] for the homogeneous case, can then be given by

$$\bar{R}_\Phi = \int_0^\infty \int_0^\infty 2\pi\lambda r \exp\left(-\pi\lambda r^2\right) \mathcal{L}_W\left(s\right) \mathcal{L}_{I_\Phi}\left(s\right) dr d\gamma, \tag{1}$$

where r is the radial distance between the user location and the BS, γ is the threshold, $s = \mu\left(e^\gamma - 1\right) r^\beta$, $\mathcal{L}_W\left(s\right)$ is the Laplace functional of the noise, and $\mathcal{L}_{I_\Phi}\left(s\right)$ is the Laplace functional of the interference. Repeated differentiation of the Laplace functional (or equivalently the Laplace transform) can be used to find the moments of the interference. Therefore, the Laplace functional of the interference can be considered to be equivalent to the distribution of the interference. Hence, the rest of this work focuses on attempts to obtain the Laplace functional of the interference.

A. Laplace Functional of the Interference

The Laplace functional of the interference can be written as

$$\begin{aligned} \mathcal{L}_{I_\Phi}\left(s\right) &= \mathbb{E}_o^!\left[\exp\left(-s\sum_{x\in\Phi} h/l(x-z)\right)\right] \\ &= \mathbb{E}_o^!\left[\prod_{x\in\Phi} \exp\left(-sh/l(x-z)\right)\right] \\ &= \mathbb{E}_o^!\left[\prod_{x\in\Phi} \mathcal{L}_h\left(s/l(x-z)\right)\right], \end{aligned}$$

where $\mathcal{L}_h\left(s/l(x-z)\right)$ is the Laplace functional of the receive power and $\mathbb{E}_o^!\left[\cdot\right]$ is the Palm expectation with respect to the *reduced* Palm distribution conditioned on the fact that there is a point of the point process at the origin. Let $\mathcal{L}_h\left(s/l(x-z)\right) = v(x-z)$, where V is a family of functions such that $v \in V$ and $0 \le v(x) \le 1, \forall\ x$, which implies $\mathcal{L}_{I_\Phi}\left(s\right) = \mathbb{E}_o^!\left[\prod_{x\in\Phi} v(x-z)\right]$. For PCPs [2], we have

$$\mathcal{L}_{I_\Phi}\left(s\right) = \mathbb{E}_o^!\left[\prod_{x\in\Phi} v(x-z)\right] = \mathbb{E}_o\left[\prod_{x\in\Phi} v(x-z)\right], \tag{2}$$

where $\mathbb{E}_o\left[\cdot\right]$ is the Palm expectation with respect to the Palm distribution. Since the Laplace functional of a PCP is difficult to compute, the generating functional is used instead.

[2] [6] and [8] show that the reduced Palm expectation of a PCP is equivalent to the Palm expectation of the PCP.

B. Generating Functional of the Interference

Let M be a set of all locally finite counting measures φ on $\mathbb{R}^d$ with a σ-algebra of Borel sets, $\mathcal{B}^d$. Let $\mathcal{M}$ be the σ-algebra generated by the sets $\left\{\varphi : \varphi \in M, \varphi(B) = k\right\}, \forall k = 0, 1, 2, \cdots$ and $B \in \mathcal{B}_0^d$, where $\mathcal{B}_0^d$ is the system of bounded sets in $\mathcal{B}^d$. From [6], the generating functional for a PCP is given by

$$G_{\Phi(B)}\left(t\right) = \mathbb{E}_o\left[\prod_{x\in\Phi} v(x)\right] = G(v),$$

for $t \in [0, 1]$ and $B \in \mathcal{B}_0^d$, with $v(x) = 1 + (t-1)\mathbf{1}_B(x)$ as per the definition in Section II-A above.

For a stationary PCP Φ with distribution P, denote the parent points (process of cluster centers) as Φ_c with distribution P_c, intensity λ_c, and intensity measure Λ_c. Denote the process of cluster members Φ_m with distribution P_m. Each point $x \in \Phi_c$ triggers a cluster member process $\Phi_m^{(x)} \sim P_m^{(x)}$ which is independent of Φ_c and $\Phi_m^{(y)}$, if $y \neq x$. This implies $P_m^x(Y) = P_m(T_x Y)$ for $Y \in \mathcal{M}$, where for any set defined as φ: $(T_x\varphi)(B) = \varphi(B + x)$. In this scenario, $\varphi(B + x)$ is the counting measure on translated sets. From [6], the reduced Palm distribution of a PCP is given by $P_0^! = P * \tilde{P}_0^!$ where $\tilde{P}$ is the KLM measure [9] and $*$ is the convolution operator. For a stationary PCP, we know that

$$\tilde{P}_0^!(Y) = \frac{1}{\lambda_m} \int\limits_M \sum_{x\in\varphi} \mathbf{1}_Y\left(T_x\varphi - \delta_0\right) P_m\left(d\varphi\right), \forall Y \in \mathcal{M},$$

where $\delta_x(B) = \mathbf{1}_B(x)$ is the Dirac measure and λ_m is the mean number of points of the representative cluster (cluster intensity). For all non-negative $(\mathcal{B}^d)^k \otimes \mathcal{M}$-measurable mappings, $q : (\mathbb{R}^d)^k \times M \longrightarrow \mathbb{R}_+^1$, such that

$$\begin{aligned} &\sum_{x_1, x_2, \cdots \in\Phi}{}^{*} q\left(x_1, x_2, \cdots, x_k, \Phi - \sum_{l=1}^k \delta_{x_l}\right) = \\ &\sum_{l=1}^k \sum_{K_1\cup\cdots\cup K_l = K} \sum_{y_1, \cdots, y_l \in \Phi_c}{}^{*} \sum_{\substack{k_1\in\Phi_m(y_1)\\ k_1\in K_1}}{}^{*} \cdots \sum_{\substack{k_l\in\Phi_m(y_l)\\ k_l\in K_l}}{}^{*} \\ &q\left(x_1, \cdots, x_k, \sum_{z\in\Phi_c - \delta_{y_1}\cdots - \delta_{y_l}} \Phi_m^{(z)} + \sum_{j=1}^l \left(\Phi_m^{(y_j)} - \sum_{k_j\in K_j} \delta_{x_{k_j}}\right)\right), \end{aligned}$$

$P-a.s.$ ($P-$almost surely), for all $k = 1, 2, \cdots$ where the sum $\sum_{K_1\cup\cdots\cup K_l = K}(\cdot)$ is taken over all partitions of the set $K = \{1, \cdots, k\}$ into 'l' disjoint non-empty subsets K_j. The sum $\sum_{y_1, \cdots, y_l\in\Phi_c}{}^{*}$ is taken over all k-tuples of pairwise distinct $y_1, \cdots, y_l \in \mathbb{R}^d$ with $\varphi(\{y_j\}) > 0, j = 1, \cdots, k$. Then, from Theorem 1 in [10],

$$\begin{aligned} &\int\limits_M \sum_{x_1, \cdots, x_k \in\varphi}{}^{*} q\left(x_1, \cdots, x_k, \varphi - \sum_{l=1}^k \delta_{x_l}\right) P(d\varphi) = \\ &\int\limits_{(\mathbb{R}^d)^k} \int\limits_M q\left(x_1, \cdots, x_k, \varphi\right) P^!_{x_1, \cdots, x_k}(d\varphi) \alpha_P^{(k)}\left(d\left(x_1, \cdots, x_k\right)\right), \end{aligned} \tag{3}$$

where $\alpha_P^{(k)}(B) = C_P^{(k)}(B \times Y)$ is the k-th order reduced Campbell measure given by

$$C_P^{(k)}(B \times Y) = \int\limits_M \sum_{x_1,\cdots,x_k \in \varphi}^{*} \mathbf{1}_B(x_1,\cdots,x_k)\, \mathbf{1}_Y\left(\varphi - \sum_{l=1}^{k} \delta_{x_l}\right) P(d\varphi).$$

The equations in the lines above that culminate in equation (3) show that the *n*-fold Palm distributions which describe a PCP can be seen as a natural generalization of the "usual" Palm distributions (or the first order Palm distributions). The first order Palm distribution is defined as the density of the Campbell measure of Φ with respect to the intensity measure. According to Campbell's Theorem [8], for a Poisson process Φ_c distributed as P_c with intensity measure Λ_c, $\alpha_{P_c}^{(k)} = \Lambda_c \times \Lambda_c \cdots \times \Lambda_c$ and $P^!_{x_1,\cdots,x_k}(Y) = P(Y)$, $\forall Y \in \mathcal{M}$.

Theorem 1 from [11] states: If Λ_c is σ-finite and $\mathbb{E}\left[(\Phi_m(\mathbb{R}^d))^k\right] < \infty$ then

$$\int\limits_{(\mathbb{R}^d)^k} \int\limits_M q(x_1,\cdots,x_k,\varphi)\, P^!_{x_1,\cdots,x_k}(d\varphi)\alpha_P^{(k)}(d(x_1,\cdots,x_k))$$
$$= \sum_{l=1}^{k} \sum_{K_1\cup\cdots\cup K_l = K} \int\limits_{(\mathbb{R}^d)^l} \int\limits_{(\mathbb{R}^d)^{|K_1|}} \cdots \int\limits_{(\mathbb{R}^d)^{|K_l|}} \int\limits_{M^{l+1}}$$
$$q\left(x_1,\cdots,x_k,\psi + \sum_{j=1}^{l}\psi_j\right) P(d\psi) \times$$
$$\prod_{j=1}^{l}\left(P_m^{(y_j)}(d\psi_j)\right)_{(x_{k_j}:k_j\in K_j)} \prod_{j=1}^{l} \alpha^{(|K_j|)}_{P_m^{(y_j)}}\left(d\left(x_{k_j}:k_j \in K_j\right)\right)$$
$$\times (\Lambda_c \times \Lambda_c \cdots \times \Lambda_c)(d(y_1,\cdots,y_l)), \tag{4}$$

where ψ is the finite representative cluster process whose reduced Palm distribution is given by $\tilde{P}^!_0$ and $\psi_y = \psi + y$. $|K_j|$ in equation (4) stands for the cardinality of the set K_j. The equation given above is a generalization of the main result from [12], where Mecke shows that a doubly stochastic Poisson process can be represented as a mixture of Poisson processes.

Using the above result for the special case of $k = 1$ and the set $K = Y(x,r) = \{\varphi : \varphi \in M, \varphi(b(x,r)) = 0\}$, where $b(x,r) = \left\{y : y \in \mathbb{R}^2, ||x-y|| \le r\right\}$, the mapping q is now representative of the family of functions V described by $v \in V, 0 \le v(x) \le 1$ ($\forall\ x$) in equation (2). Therefore, the generating functional is then given by

$$G(v) = \int\limits_M \prod_{x\in\varphi} P_m(Y(x,r))\, \Lambda_c(d\varphi) \times \frac{1}{\lambda_m}\int\limits_{\mathbb{R}^2} P^!_{m(x)}(Y(x,r))\, \alpha_m^{(1)}(dx),$$

where $\alpha_m^{(1)}(dx)$ is the first order reduced Campbell measure. By definition, a PCP is a process where in the parent points are driven by a Poisson process. This implies

$$G(v) = \exp\left\{-\lambda_c \int\limits_{\mathbb{R}^2} [1 - P_m(Y(x,r))]\, dx\right\} \times \frac{1}{\lambda_m}\int\limits_{\mathbb{R}^2} P^!_{m(x)}(Y(x,r))\, \alpha_m^{(1)}(dx).$$

Considering a Neyman-Scott process [6], where Φ_m consists of points which are independently and identically distributed about the origin according to some distribution $f(x)$ with a cluster size N, which has the probability generating functional $g(a) = \sum_{n\ge 0} P(N = n)\, a^n$. So we have

$$\begin{aligned} P_m(Y(x,r)) &= \sum_{n=0}^{\infty} P(N = \Phi_m(b^c(x,r)) = n) \\ &= g(F(b^c(x,r))), \\ P^!_{m(x)}(Y(x,r)) &= (\mathbb{E}(N))^{-1} g'(F(b^c(x,r))), \\ \alpha_m^{(1)}(dx) &= \mathbb{E}(N)F(B), \qquad \forall B \in \mathcal{B}^2 \end{aligned}$$

where $g'(\cdot)$ is the derivative of $g(\cdot)$, $b^c(x,r) = \mathbb{R}^2 \setminus b(x,r)$, and $F(B) = \int_B f(x)dx$. Substituting the above results in a generating functional of the form

$$G(v) = \exp\left\{-\lambda_c \int\limits_{\mathbb{R}^2} [1 - g(F(b^c(x,r)))]\, dx\right\} \times \frac{1}{\lambda_m}\int\limits_{\mathbb{R}^2} g'(F(b^c(x,r)))\, f(x)dx.$$

The generating functional for a Poisson point process is $g(F(b^c(x,r))) = \exp\{-\lambda_m[1 - F(b^c(x,r))]\}$, which implies $g'(F(b^c(x,r))) = \lambda_m[\exp\{-\lambda_m[1 - F(b^c(x,r))]\}]$. Hence the generating functional is given by

$$G(v) = \exp\left\{-\lambda_c \int\limits_{\mathbb{R}^2} [1 - \exp\{-\lambda_m[1 - F(b^c(x,r))]\}]\, dx\right\} \times \int\limits_{\mathbb{R}^2} \exp\{-\lambda_m[1 - F(b^c(x,r))]\}\, f(x)dx,$$

$$\Longrightarrow G(v) = \exp\left\{-\lambda_c \int\limits_{\mathbb{R}^2} [1 - \exp\{-\lambda_m F(b(x,r))\}]\, dx\right\} \times \int\limits_{\mathbb{R}^2} \exp\{-\lambda_m F(b(x,r))\}\, f(x)dx.$$

Substitution of the series approximation of $\exp(-y)$ as

$\exp(-y) = 1 - y + \mathcal{O}(y^2)$, and simplification results in

$$G(v) \approx \exp\left\{-\lambda_c \int_{\mathbb{R}^2} \lambda_m F\left(b(x,r)\right)\right\} dx \times \left[1 - \lambda_m \int_{\mathbb{R}^2} F\left(b(x,r)\right) f(x) dx\right]. \quad (5)$$

In the case of a heterogeneous network that is represented by a PCP, then

$$F\left(b(x,r)\right) = \int_{\mathbb{R}^2} \mathcal{L}_h\left(s/l(x-y)\right) f(y) dy.$$

If the fading h is considered to be exponentially distributed with mean μ, the Laplace transformation of the receive power is $\mathcal{L}_h\left(s/l(x-y)\right) = \frac{\mu}{\mu + \frac{\mu t' l(x-y)}{l(r')}}$ where the radial distance between the user location and a point of PCP to which it is connected is r'. This substitution in equation (5) results in

$$G\left(v\right) \approx \exp\left\{-\lambda_c \lambda_m \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{1 + \frac{t' l(x-y)}{l(r')}} f(y) dy dx\right\} \times \left[1 - \lambda_m \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{1 + \frac{t' l(x-y)}{l(r')}} f(y) f(x) dy dx\right].$$

Therefore,[3]

$$\mathcal{L}_{I_\Phi}\left(s\right) \approx \exp\left\{-\lambda_c \lambda_m \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{1 + \frac{t' l(x-y-z)}{l(r')}} f(y) dy dx\right\} \times \left[1 - \lambda_m \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{1 + \frac{t' l(x-y-z)}{l(r')}} f(y) f(x) dy dx\right]. \quad (6)$$

Hence, equation (6) gives a complete description of the interference in a heterogeneous network that consists of micro BSs clustered around macro BSs. The expression derived above comprises of the interference experienced by a user at a given point from all BSs within (except from the BS that it is connected to) and outside the cluster to which it is connected to. Inter-cluster interference is the interference experienced at a given location from all the BSs in the area. Intra-cluster interference is the interference experienced at a given location from other BSs within the cluster. The first term of the product on the RHS of equation (6), the exponential term, describes the inter-cluster interference and the second term of the product describes the intra-cluster interference.

Since equation (6) essentially involves a few convolution operations, obtaining a closed form solution is unlikely in most cases. Our aim is to be able to characterize the interference as succinctly as possible. Therefore, it is essential to use other methods to understand the behavior of the interference. Our method of choice is to observe the asymptotic behavior of $\mathcal{L}_{I_\Phi}(s)$.

C. Asymptotic Behavior of $\mathcal{L}_{I_\Phi}(s)$

Let $(W_n)_{n \geq 1}$ be a sequence of compact sampling windows in $\mathbb{R}^d$. Let the eroded sets, $D_n := \bigcap_{x \in b(0,r)} (W_n + x), n \geq 1$ satisfy the *Regularity Condition* [4].

Regularity Condition: There exists a sequence of d-dimensional rectangles, $A_n = [0, a_n^{(1)}] \times \cdots \times [0, a_n^{(d)}]$ and constants $C_1, C_2 > 0$ such that:

(a) $\nu(A_n) \xrightarrow[n\to\infty]{} \infty$, and $\min_{1 \leq i \leq d} a_n^{(i)} \geq C_1$,

(b) $D_n \subseteq A_n$, and $\nu(D_n) \geq C_2 \nu(A_n)$ for every $n \geq 1$,

where $\nu(\cdot)$ is the Lebesgue measure.

Let $S(r)$ be the unbiased estimator of $\mathcal{L}_{I_\Phi}(s)$. Define $S(r) = \lambda_p G(v(x-z))$, where $0 < \lambda_p < \infty$ and $\lambda_p = \mathbb{E}\left[\Phi\left([0,1]^d\right)\right]$ is the intensity of the PCP Φ in the closed interval considered. For stationary PCPs, $\lambda_p = \lambda_c \lambda_m$. Over the eroded sets, $S(r)$ is defined by

$$S_n(r) = \frac{1}{\nu(D_n)} \sum_{x \in \Phi} \mathbf{1}_{D_n}(x) \mathbf{1}_{Y(x,r)}(\Phi - \delta_x),$$

where $\nu(D_n)$ is the Lebesgue measure of D_n. The asymptotic behavior and unbiasedness of a class of estimators for stationary PCPs under a strong mixing [5] condition known as the Brillinger mixing or B-mixing [6] is shown in [13]. If the B-mixing condition is satisfied, a random variable can be constructed from an unbiased estimator (for ex: $S_n(r)$ as defined above). This condition has been shown to be true in [13]. Therefore, a centered random variable $Z_n(r)$ can be defined as

$$Z_n(r) = \left(\nu(D_n)\right)^{1/2} \left(S_n(r) - \mathbb{E}\left[S_n(r)\right]\right).$$

Theorem. *For a radius* $r \geq 0$, *if* $\lim_{n\to\infty} Var\left[Z_n(r)\right] = \sigma^2_{\lambda_p}(r) > 0$, *then*

$$Z_n(r) \xrightarrow[n\to\infty]{D} N(0, \sigma^2_{\lambda_p}(r)) \quad (7)$$

where $N(0, \sigma^2_{\lambda_p}(r))$ *is a Gaussian distribution.*

Proof: The assertion in equation (7) can be proved along the lines of Theorem 4 in [11]. Introduce a truncated PCP, Φ_ρ whose cluster center process is still $\Phi_c \sim P_c$ but the process

[3]Note: $\mathcal{L}_{I_\Phi}(s) = G(v(x-z))$. Equation (6) is obtained after 's' has been substituted with its original value (i.e $\mu\left(e^\gamma - 1\right) r^\beta$). '$(e^\gamma - 1)$' has been replaced by '$t'$' for brevity.

[4]This condition needs to be satisfied in order for us to be able to obtain a Lebesgue measure on $\mathbb{R}^d$. The regularity condition enables treating the Lebesgue measure on $\mathbb{R}^d$ as a product measure which is constructed from premeasures defined on the rectangles in $\mathbb{R}^d$. The premeasures help obtain an outer measure on $\mathbb{R}^d$ and Carathéodory's Theorem is then used to obtain the Lebesgue measure.

[5]Strong mixing implies that for any two realizations of the random variable, given a sufficient amount of time between the two realizations, their occurrence is independent.

[6]A brief definition, along with the sufficient conditions for point processes to be B-mixing, is also given in [13].

of cluster members $\Phi_{m\rho}$ consists of atoms of Φ_m which are located in the sphere $b(0,\rho)$, where $\rho > r$; i.e. $\Phi_{m\rho}(\{x\}) > 0$ if $\Phi_m(\{x\}) > 0$ and $||x|| \leq \rho$. For $A \in \mathcal{B}_0^d$,

$$\begin{aligned} S_{n\rho}(r,A) &= \frac{1}{\nu(D_n)} \sum_{x\in\Phi_\rho} \mathbf{1}_{A\cap D_n}(x)\mathbf{1}_{Y(x,r)}(\Phi_\rho - \delta_x) \\ &= \frac{1}{\nu(D_n)} \sum_{y\in\Phi_m} \sum_{x\in\Phi_m^{(y)}} \mathbf{1}_{A\cap D_n \cap b(y-\rho)}(x)\mathbf{1}_{Y(x,r)}(\Phi_{m\rho}^{(y)} - \delta_x) \\ &\quad \times \prod_{z\in\Phi_c - \delta_y} \mathbf{1}_{Y(x,r)}(\Phi_{m\rho}^{(z)}), \end{aligned}$$

which implies that the centered random variable can be written as $Z_{n\rho}(r) = (\nu(D_n))^{1/2} (S_{n\rho}(r,D_n) - \mathbb{E}[S_{n\rho}(r,D_n)])$. By the definition of Φ_ρ, the random variables $S_{n\rho}(r,A)$ and $S_{n\rho}(r,B)$ are independent if the sets A and B are separated greater than $2(\rho + r)$. Define a set $E_z = [z_1 - 1, z_1) \times \cdots \times [z_d - 1, z_d)$ for any $\{z \in U_n \subset Z^d\}$ where $Z^d = \{z = (z_1,\cdots,z_d) : z_i = 0, \pm1, \pm2, \cdots ; i = 1,\cdots,d\}$ and $U_n = \times_{i=1}^{d} \{1,2,\cdots,[a_n^{(i)}]+1\}$. Consider a family of random variables

$$\begin{aligned} X_{nz}(r) &= \frac{(\nu(D_n))^{1/2} (S_{n\rho}(r,E_z) - \mathbb{E}[S_{n\rho}(r,E_z)])}{\text{Var}[Z_{n\rho}(r)]} \\ &= \frac{Z_{n\rho}(r)}{\text{Var}[Z_{n\rho}(r)]}. \end{aligned}$$

Consequently, $X_{nz}(r)$ form an m-dependent random field. Therefore, as in Theorem 1 of [14], it can be proven that

$$\frac{Z_{n\rho}(r)}{\text{Var}[Z_{n\rho}(r)]} \xrightarrow[n\to\infty]{D} N(0,1), \quad (8)$$

since X_{nz} satisfies the following conditions for every fixed $\epsilon > 0$:

(i) $\sum_{z\in U_n} \mathbb{P}(|X_{nz}| \geq \epsilon) \xrightarrow[n\to\infty]{} 0$,

(ii) $\sum_{z\in U_n} \mathbb{E}[X_{nz}^2(\epsilon)] \leq C(\epsilon) < \infty$,

(iii) $\mathbb{E}[S_n(\epsilon)] \xrightarrow[n\to\infty]{} a \in \mathbb{R}$ and $\text{Var}[S_n(\epsilon)] \xrightarrow[n\to\infty]{} \sigma^2$,

where $C(\epsilon)$ is a positive constant that changes with ϵ and $\sigma > 0$. The above conditions are shown to be true in [14] and the references therein. Then it remains to be shown that

$$\lim_{\rho\to\infty} \sup_{n\geq1} \text{Var}(Z_n(r) - Z_{n\rho}(r)) = 0, \qquad \forall r \geq 0. \quad (9)$$

By definition,

$$\begin{aligned} \text{Var}(Z_n(r) - Z_{n\rho}(r)) &= \mathbb{E}(Z_n(r))(Z_n(r) - Z_{n\rho}(r)) - \\ &\quad \mathbb{E}(Z_{n\rho}(r))(Z_n(r) - Z_{n\rho}(r)). \end{aligned}$$

From the equation above, it can then be shown that

$$\lim_{\rho\to\infty} \sup_{n\geq1} \mathbb{E}(Z_n(r))(Z_n(r) - Z_{n\rho}(r)) = 0,$$

and

$$\lim_{\rho\to\infty} \sup_{n\geq1} \mathbb{E}(Z_{n\rho}(r))(Z_n(r) - Z_{n\rho}(r)) = 0$$

as in [11]. For the functional limit theorem to hold, the tightness of Z_n must be proven; i.e. to prove the convergence of $Z_n = (Z_n(r), 0 \leq r \leq R)$ with $n = 1,2,\cdots$ as a sequence of $D[0,R]$-valued random elements, the fourth moments of the increments $Z_n(t) - Z_n(s), 0 \leq s \leq t \leq R$ need to be bounded. This can be done by determining bounds on $\mathbb{E}(Z_n(t) - Z_n(s))^4$ by means of the fourth- and second-order cumulants[7] of $(\nu(D_n))^{1/2}(S_n(t) - S_n(s))$ as in Lemma 2 of [11]. The bounds are given by

$$\mathbb{E}(Z_n(t) - Z_n(s))^4 \leq C_1 \left[(t-s)/\nu(D_n) + (t-s)^2\right],$$

for a constant $C_1 > 0$. Therefore, from equations (8) and (9),

$$Z_n(r) \xrightarrow[n\to\infty]{D} N(0, \sigma_{\lambda_p}^2(r)).$$

■

Therefore, the estimator of the interference over a normalized area can be approximated by a Gaussian random variable with zero mean and variance $\sigma_{\lambda_p}^2(r)$. Since the estimator is a linear function of the generating functional of the interference, the interference itself can be approximated by a Gaussian random variable with zero mean and variance $\sigma_{\lambda_p}^2(r)/\lambda_p^2$.

Significance of the findings: The above proof shows that the interference for heterogeneous networks can be assumed to have a Gaussian distribution. A salient feature of this result is that it holds irrespective of the deployment topology (i.e. it is independent of the location of macro BSs and micro BSs in the area). It is also important to note that the description of the interference as a Gaussian random variable remains unchanged based on the definition of functionals such as transmit power, fading, pathloss, etc. that are attributed to these points. This result, therefore, implicitly holds for all pathloss exponents $\beta > 2$. The influence of the functionals can be observed only in the variance of interference. Hence, well behaved functionals that can describe the pathloss, fading, etc. can assist in achieving a complete and accurate description of the interference.

III. Conclusions and Future Work

This work examines the interference in a HetNet consisting of macro BSs and micro BSs. The formulation consists of micro BSs clustered around a macro BS according to a given distribution dependent on the user density in a given area. Functionals representing various system specifications such as transmit power, fading, etc. act on every point of the point process. There are two main results that have been expounded in this paper. The first is an expression for the generating functional of the interference, which is analogous to finding the density function of the interference, and hence completely describes its behavior in a given area. The second is a proof that examines the asymptotic behavior, and shows that the

[7] The cumulant is a quantity that provides an alternative to the moments of a function. The cumulant generating functional can be expressed as the logarithm of the moment generating functional.

interference can be described by a Gaussian random variable whose variance is dependent on the functionals acting on each point of the point process.

It should be emphasized that the results chronicled in this paper are independent of the network topology (or the precise locations of macro BSs and micro BSs). While the first result (the generating functional) is dependent on the distribution of micro BSs around the macro BS, the second result (the theorem) is independent of the distribution. The convergence of the distribution of the interference to a Gaussian distribution is also independent of the functionals acting on points of the point process. However, this does not imply that the functionals do not have any impact on the distribution of the interference. It is worth reiterating that, the variance of the distribution of the interference depends on the functionals chosen to describe transmit power, fading, pathloss, etc. These results form a significant step towards improving the authors' understanding of HetNets.

As mentioned in the latter part of Section I, this work is a part of the authors' endeavors to estimate the energy consumption of a heterogeneous network based on a capacity constraint. Our future work consists of efforts to achieve this goal. The first step would be to find an expression for the variance of the distribution to which the convergence of the interference has been demonstrated. Once this is achieved, we also intend to corroborate these findings using simulations. A comparison of our theoretic frame work with simulations will indubitably help improve and refine our models.

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