Pyramids in Logo: A School Project in 'Search' of the Fourth Dimension

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Abstract: In this paper we present a school project where students constructed three-dimensional pyramids using the Logo programming language, complemented with paper-and-pencil, dynamic geometry (Cabri) and spreadsheet (Excel) investigations. The aim of this project was to give, through a fun and meaningful way, and using a *constructionist* approach, junior secondary students (12-14 year olds), early access to advanced topics such the applications of the Pythagorean Theorem and of trigonometric functions, as well as three-dimensional work, while at the same time covering one of the themes included in the curriculum for this age-group (the pyramid).

Keywords: School project, three dimensions, mathematics, trigonometry, computer programming, Logo, Cabri-Géomètre

1. Introduction and Background

Since 2001-02, we have been working in our mathematics classrooms with the materials and digital tools provided by a government-sponsored national program: the Teaching Mathematics with Technology program (EMAT).

1.1 The EMAT Program and its Implementation in Our Schools

EMAT is a program that was sponsored, beginning in 1997, by the Mexican Ministry of Education to promote the use of new technologies, using a constructivist approach, to enrich and improve the current teaching and learning of junior secondary mathematics in Mexico. A study [1] carried out in Mexico and England involving mathematical practices in science classes, revealed that in Mexico few students were able to close the gap between the formal treatment of the curricular topics and their possible applications. This suggested that it was necessary to replace the formal approach of the then official curriculum, with a “down-up” approach capable of fostering the students’ explorative, manipulative, and communication skills. Thus, a major part of the EMAT program is a pedagogical model that emphasizes exploratory and collaborative learning. The main computer tools of the EMAT program are spreadsheets (Excel), Dynamic Geometry (Cabri-Géomètre), and the Logo (MSWLogo) programming language. These pieces of software were chosen [2] on the criteria that they would be open tools; that is, where the user could be in control and have the power of deciding how to use the software, and flexible enough so that they could be used with different didactical objectives.

At the beginning we used these tools independently, covering different themes with each of them. Logo was the last tool that we incorporated into our schools, and one of the immediate things we noticed was how much it enriched both children’s motivations for exploring mathematical topics and also the use of the other tools. Because of the programming experience with Logo, students began asking if it was possible to also program the other tools. This led us, for example, to show them how to create macros in Excel and Cabri.

We now believe in the importance of using in an integrated and complementary way, a variety of tools for learning, since we consider that each tool brings with it a different type of knowledge and constitutes a different epistemological domain [3]. We also believe in the importance of constructionist [4] or programming activities for meaningful learning. We have in fact observed that students who otherwise have difficulties in learning, exhibit other kinds of abilities through these computer-based approaches to mathematics teaching and learning.

1.2 From Isolated Activities to Long-term Projects: The “Painless Trigonometry” Project

In recent years we have tried to develop long-term projects that incorporate powerful tools, and also serve as a means to introduce students to topics of mathematics that are normally considered too advanced for them (such as trigonometry). We have realized that through these projects we can cover, in a fun and meaningful way, most of the topic in the compulsory curriculum.

In particular, in the academic year 2005-06 we developed a long-term trigonometry project called “Painless Trigonometry” [5] for introducing young students to the Pythagorean theorem, basic trigonometry concepts, and their applications using explorations and constructive activities with Cabri-Géomètre, Excel and Logo. Through trigonometry, we covered other mathematical topics in the curriculum such as addition, subtraction, multiplication and division; powers and square roots of whole and rational numbers; algebra (including constants and variables). Approximately 250 students of 12-14 years of age participated, in the project in that first year, in 6 groups of the first two grades of two junior secondary schools in Mexico. Students thoroughly enjoyed the activities and gained interest in mathematics. They also developed problem-solving and collaborative skills. Furthermore, in written tests after the project, the students showed an understanding of the “advanced” trigonometry concepts, as well as of other algebraic ideas. Through the programming activities in Logo, students also learned to work with three-dimensional elements and animations.

The project taught us that mathematics can be learned through different tools and ways to traditional ones, that one can go further than usual standards, and that students can learn to see the world of mathematics in more than two dimensions.

2. "In Search of the Fourth Dimension, While in Three": A New School Project

In the academic year 2007-2008, a question came up when our junior secondary first-year students (12-13 year-olds) were learning Logo: "Is it possible to work in four dimensions in Logo?" The reply was that we could "search" for the fourth dimension but we would work and learn in three dimensions; and that is what gave rise to the idea of a new school project for working in three dimensions, that was named for fun: "In search of the fourth dimension, while in three". We then needed to find a curricular topic for junior secondary grades 1 and 2 that could be worked in three dimensions, so we chose the pyramid.

We started playing with paper-and-pencil in a geometry game to draw triangles and squares and whatever else was needed for a pyramid. We then transferred that activity to doing it with dynamic geometry (in Cabri-Géomètre) and used an Excel spreadsheet to help us in computing areas and perimeters. In the end, we programmed a pyramid in Logo. Although not the same students as in 2006-2007, we worked in this new project with approximately the same number of participants (between the ages of 12 and 14), as we did in the "Painless Trigonometry" project.

2.1 Laying the Foundations for the Construction of a Pyramid

After the paper-and-pencil activities mentioned above, we had students constructing triangles (of different shapes and sizes) and regular polygons in both Cabri and Logo, with the construction and epistemological difference that each software-environment implies (in Logo, the construction is more linear). The programming of the basic shapes, not only served as the foundation for the pyramid, but also helped children gain experience in programming. Below are the sample basic procedures that students built in Logo:

```
to square                    to triangle
repeat 4 [fd 100 rt 90]      repeat 3 [fd 100 rt 360/3]
end                          end
```

In order to facilitate the construction of the pyramid the students first constructed concrete models with wooden sticks, which helped them visualize what they had to do on the screen. Then, students attempted to construct a pyramid in Cabri, but they were only able to do an isometric projection (Fig. 1).

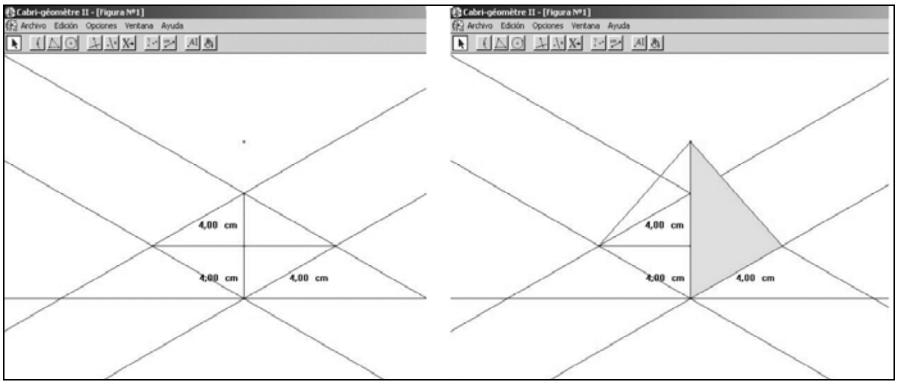

Fig. 1 Construction process of the pyramid's isometric projection in Cabri.

2.2 From Two to Three Dimensions

In Logo, students can work in three-dimensional mode (using the perspective command in MSWLogo). They began working in this mode by constructing a cube, Sometimes they lost the sense of direction because they lacked a reference. But the project had been previously enriched by an interaction with a fellow teacher to whom we were teaching Logo. This teacher felt disoriented when moving the turtle around the screen. It occurred to us that if that could happen to some people in two dimensions, it would be more common in three. So as an aid we added the task in our project for students to construct a system of axes in three dimensions; that is, a system with an x-axis, a y-axis and a z-axis. The construction of a system of axes helped them greatly to find their orientation:

```
to axis
setpencolor 2
forward 150 back 300 forward 150
setpencolor 4
right 90
forward 150 back 300 forward 150
left 90
setpencolor 5
downpitch 90
forward 150 back 300 forward 150
end
```

Once they had a set of axes, they constructed cubes (Fig. 2) by joining squares, going forward then left, forward then right, then down and left, and then played with those cubes (Fig. 3).

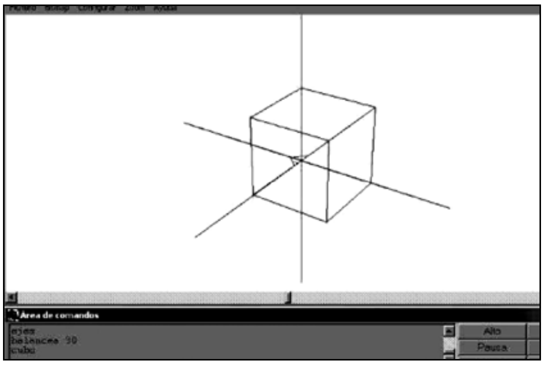

Fig. 2 Construction in 3 dimensions of the cube using a set of axis

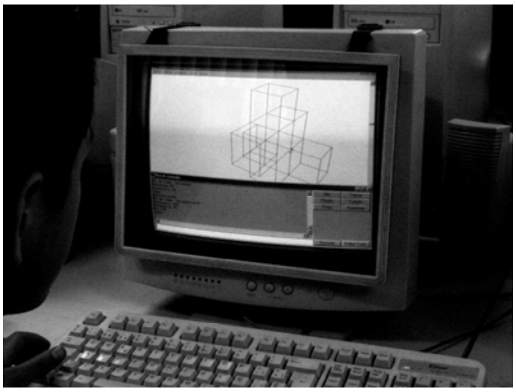

Fig. 3 Playing with cubes before moving on to the Pyramid task

2.3 The Pyramid Task

After constructing a cube the children felt ready to tackle the objective: building a pyramid. For this, in the classroom discussions on how to proceed it was felt that the center of the base-square of the pyramid was needed as a reference for the tip of the pyramid. For this, they used Cabri to help them in their visualization of the figure.

They then discovered that in Logo they could use Pythagoras Theorem to draw the inner diagonals. Thus, in order to construct the pyramid in Logo, the students required the use of trigonometric ideas and functions. So, for this project, we also used the activities of the "Painless Trigonometry" project. This meant that this project was even longer and more complex, but we carried forward because the students were highly motivated. Using Pythagoras Theorem, students were able to construct right triangles in Logo (Fig. 4):

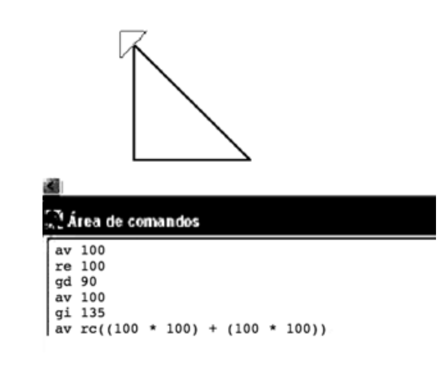

Fig. 4. A right triangle in Logo

```
to righttriangle
forward 100 back 100
right 90
forward 100
left 135
forward sqrt((100 * 100)
 + (100 * 100))
end
```

They even wrote procedures for drawing a generic right triangle (which some called `pythagoras`), and for which they had to also use the `arctan` function for calculating the angle. We found this particularly interesting since these were 12-13 year-old children and that kind of mathematical application of trigonometry is only seen in our country after the age of 15 or 16. But we observed that the children acted in this as experts, in their quest to reach the goal of the “game”. On our side we didn’t push them, we let them progress at their own pace, letting them be the discoverers of their constructions.

```
to pythagoras :a :b
forward :a back :a
right 90
forward :b
left 180
right arctan (:a/:b)
forward sqrt (:a*:a)+(:b*:b)
end
```

Using the knowledge from the right triangle experience, the students moved to constructing a pyramid. They used their procedures for the axis and for the square, as well as Pythagoras Theorem to find the center of the base-square and height of the pyramid. For a base-square of size 100, the center of the square is located at half the diagonal that is at a distance of 70.71067811865 (= sqrt(100 *100 + 100 *100))/2 = $100\sqrt{2} / 2$). For the height of the pyramid they used the same distance as half of the diagonal of the base-square. In this way they were able to reach the upper tip of the pyramid (Fig. 5).

```
to pyramid
square
right 45
forward (sqrt(100 *100 + 100 *100))/2
downpitch -90
forward (sqrt(100 *100 + 100 *100))/2
downpitch 135
end
```

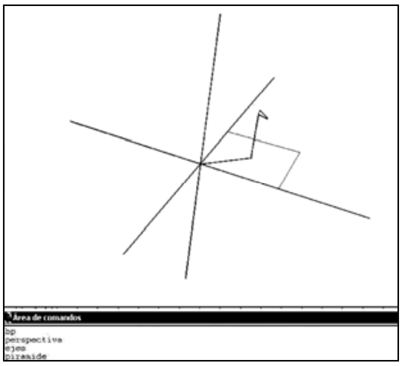

Fig. 5 Placing Logo's turtle at the tip of the pyramid-to-be, above the center of the base-square

The next step was to join the vertices of the square with the tip of the pyramid. They used again Pythagoras Theorem to find the length from the tip of the pyramid to the vertices of the base-square (the hypotenuse of a right isosceles triangle of side 70.7106781186548).

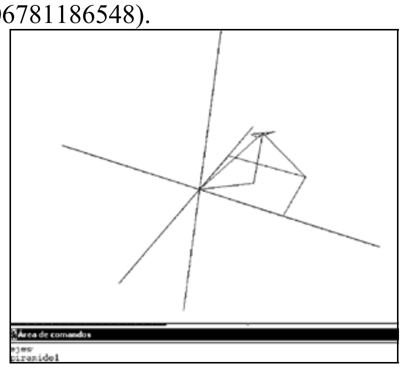

Fig. 6 The completed pyramid with the reference axis-system

The completed procedure for constructing the pyramid (Fig. 6) was something as follows[33]:

```
to pyramid
square
right 45
forward (sqrt(100 *100 + 100 *100))/2
downpitch -90
forward (sqrt(100 *100 + 100 *100))/2
downpitch 135
forward sqrt(70.7106781186548 * 70.7106781186548 +
 70.7106781186548 * 70.7106781186548)
```

[33] Please note that all the programs presented in this paper are sample procedures written by students. They illustrate their ways of thinking of the problem. Thus, they may not be examples of the most efficient ways of programming the figures in question.

```
back sqrt( 70.7106781186548 * 70.7106781186548 +
 70.7106781186548 * 70.7106781186548)
downpitch -270
forward sqrt(70.7106781186548 * 70.7106781186548 +
 70.7106781186548 * 70.7106781186548)
back sqrt(70.7106781186548 * 70.7106781186548 +
 70.7106781186548 * 70.7106781186548)
downpitch 135
rightroll 90
downpitch 135
forward sqrt(70.7106781186548 * 70.7106781186548 +
 70.7106781186548 * 70.7106781186548)
back sqrt( 70.7106781186548 * 70.7106781186548 +
70.7106781186548 * 70.7106781186548)
downpitch -270
forward sqrt(70.7106781186548 * 70.7106781186548 +
 70.7106781186548 * 70.7106781186548)
back sqrt(70.7106781186548 * 70.7106781186548 +
 70.7106781186548 * 70.7106781186548)
downpitch 135
back (sqrt(100 *100 + 100 *100))/2
rightroll -90
downpitch -90
forward (sqrt(100 *100 + 100 *100))/2
downpitch 90
rightroll 45
downpitch 90
end
```

The last step was to change the height, so they needed to use the tangent trigonometric function in order to find the necessary turning angles.

2.4 From Static Images to 3D Animation

A few months after we had finished the project, some observers came to the school and we decided to show them what we had done in that project. We assumed that the children would have forgotten much of what they had done, but to our surprise they were able to rebuild everything they had originally done over the course of many weeks, in a single 50 min. session! Then one of the observers asked if it was possible to rotate the figure in three dimensions to see it from different perspectives. In the short time available, many of the children were able to animate the figure with respect to the vertical axis, by turning the starting point and redrawing the figure several times over. This was quite a surprise for us, and a pleasant reminder that students' potential should never be underestimated.

3. Discussion and Concluding Remarks

Our school project "In search of the fourth dimension while in the third" had as aims to introduce, though a *constructionist* approach [4], junior secondary students to the applications of the Pythagorean Theorem and of trigonometric functions, while at the same time covering some of the themes included in the curriculum for this age-group, such as the study of the pyramid, but going beyond it (e.g. into three dimensions). The project was time consuming and lasted several months, but we consider it a worthwhile investment. Projects such as this one not only deal with the topic they are designed for, but also develop a need for the use of many other mathematical concepts. We are reminded of Papert's vision in his book *Mindstorms* [6] when he described the gears of his childhood.

One thing that we observed was that the students treated these computer-based projects as challenging games. This meant that they never wanted the sessions to end, and students who are shy and withdrawn in other classes and environments began to express brilliant and clever ideas which filled them with fulfillment of their self-discovery abilities and of their learning. Other students who otherwise did not engage in mathematical thinking, suddenly became leaders in this project, taking the initiative to present and "teach" to the whole class their advances and imaginative ideas. This is something that happens mostly with the Logo programming activities. Students assume the role of teachers to show their classmates how they solved the problems that emerge. The whole project was fun and motivating for students. And some students even had the opportunity to present this work at National forums.

We gave the children freedom to explore their own ideas and follow their own path. But this does not mean that we left the children on their own, we supported them but let them construct their own intellectual paths and structures.

Our students are now able to perceive mathematics as something meaningful, something that they can apply for solving a project; instead of something boring, meaningless and forced upon them –as is so often the case. It's not just giving the students access to a computer; it is the way of doing things (in our case the constructionist approach), which can awaken the creativity inside the students. As Kofi Annan expressed it [7]:

> *"[It] is not just a matter of giving a laptop to each child, as if bestowing on them some magical charm. The magic lies within—within each child, within each scientist—, scholar—, or just-plain-citizen-in-the-making."*

Logo, in particular, is an invaluable environment for this, a tool for constructing, developing abilities, and learning how to think. Some people have put this tool aside considering it a relic of times past. But amongst the sophistication of much modern software, our students still like Logo best. When we asked one of 13 year-old student why he liked Logo so much as opposed to other more modern software, he said "because [in Logo] I can express myself… [whereas] I think buttons make human beings obsolete" (!).

On the other hand, we realize also that these types of projects are challenging for educational institutions, and can be hard to accept by the educational community. At the time of this school project, we were involved in the writing of a mathematics textbook. We wanted to include activities in three dimensions, but some of our co-authors felt it would be too difficult for both teachers and students. It saddens us that student abilities are often underestimated. As was shown with the example in this paper, where students had to engage trigonometric knowledge as well as think in a three-dimensional way (going beyond the usual two-dimensional thinking of school mathematics), the computer can act as *scaffolding* [8] to give early access to powerful ideas and advanced topics.

In the words of Seymour Papert [9]:

> *"Opportunity means more than just "access" to computers. It means an intellectual culture in which individual projects are encouraged and contact with powerful ideas is facilitated." (p. xv).*

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