Terminating Tableaux for $\mathcal{SOQ}$ with Number Restrictions on Transitive Roles*

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Abstract. We show that the description logic $\mathcal{SOQ}$ with number restrictions on transitive roles is decidable by a terminating tableau calculus. The language decided by the calculus includes the universal role, which allows us to internalize TBox axioms. Termination of the system is achieved through pattern-based blocking.

1 Introduction

Number restrictions on roles are an expressive feature of description logics that allows to impose counting constraints on the number of objects that are related via a certain role. Qualified number restrictions [6] correspond to graded modalities [4, 3, 5] in modal logics. Transitive roles are prominently used in description logics for representing parthood relationships [21].

Efficient tableau algorithms are available for a wide range of description logics, including logics that contain both transitive roles and number restrictions, such as $\mathcal{SIN}$ [11], $\mathcal{SHIF}$ [8, 13], $\mathcal{SHIQ}$ [12], $\mathcal{SHOQ}$ [9], $\mathcal{SHOIQ}$ [10], and $\mathcal{SROIQ}$ [7]. In all cases, however, the language is restricted to contain no number restrictions on *complex roles*, e.g., on transitive roles, or roles containing transitive subroles. Although desirable for applications [19], number restrictions on complex roles lead to undecidability for logics extending $\mathcal{SHIN}$ [13]. In the absence of inverse roles ($\mathcal{I}$), however, the limitation of number restrictions to simple roles can be significantly relaxed [19]. In particular, the result in [19] implies the decidability of $\mathcal{SQ}$ extended by number restrictions on transitive roles. Obtained via a small model theorem, this decidability result does not yield practical decision procedures. Nor does it imply the decidability of extensions of $\mathcal{SQ}$ with nominals.

We consider the logic $\mathcal{SOQ}$ with number restrictions on transitive roles, and call it $\mathcal{SOQ}^+$. As indicated by its name, $\mathcal{SOQ}^+$ extends the basic description logic $\mathcal{ALC}$ [23] by primitive transitive roles ($\mathcal{S}$), nominals ($\mathcal{O}$), and qualified number restrictions ($\mathcal{Q}$), where we allow such restrictions on transitive roles (+). We show that reasoning in $\mathcal{SOQ}^+$ is decidable by giving a terminating tableau calculus for concept satisfiability in $\mathcal{SOQ}^+$ extended by the universal role. Having the universal role in the language allows us to internalize terminological axioms, reducing reasoning with respect to TBoxes to concept satisfiability [1, 22].

* A preliminary version of this work appeared in [17].

For termination, our calculus employs *pattern-based blocking*. Pattern-based blocking is introduced in [15, 16] for converse-free hybrid logic with global modalities. In [14], the technique is extended to graded logics subsuming $\mathcal{SOQ}$ and $\mathcal{SHOQ}$. To provide a complete treatment of number restrictions on transitive roles, we extend pattern-based blocking further, incorporating ideas [25, 2] used in tableau systems for propositional dynamic logic and propositional μ-calculus.

2 Preliminaries

Following [15, 16, 14], our formal presentation is based on simple type theory. Notationally, our presentation is based on modal syntax, but can easily be translated to the traditional DL notation [22]. We start with two base types B and I. The interpretation of B is fixed and consists of the two truth values. The interpretation of I is a nonempty set whose elements are called *individuals*. Given two types σ and τ, the *functional type* $\sigma\tau$ is interpreted as the set of all total functions from the interpretation of σ to that of τ. We write $\sigma_1\sigma_2\sigma_3$ for $\sigma_1(\sigma_2\sigma_3)$.

We employ three kinds of variables: *Nominals* x, y, z of type I (we assume there are infinitely many nominals), *propositional variables* p, q of type IB, and *role variables* r of type IIB. Since the language in question contains no role expressions other than role variables, we call role variables *roles* for short. We use the logical constants $\bot, \top : \mathrm{B}$, $\neg : \mathrm{BB}$, $\vee, \wedge, \rightarrow : \mathrm{BBB}$, $\doteq : \mathrm{IIB}$, $\exists, \forall : (\mathrm{IB})\mathrm{B}$. Terms are defined as usual. We write st for applications, $\lambda x.s$ for abstractions, and $s_1s_2s_3$ for $(s_1s_2)s_3$. We also use infix notation, e.g., $s \wedge t$ for $(\wedge)st$.

Terms of type B are called *formulas*. We employ some common notational conventions: $\exists x.s$ for $\exists(\lambda x.s)$, $\forall x.s$ for $\forall(\lambda x.s)$, and $x \dot{\neq} y$ for $\neg(x \doteq y)$.

Let us write $\exists X.s$ for $\exists x_1 \ldots x_n.s$ if $|X| = n$ and $X = \{x_1, \ldots, x_n\}$. Also, given a set X of nominals, we use the following abbreviation:

$$DX \;:=\; \bigwedge_{\substack{x,y \in X \\ x \neq y}} x \dot{\neq} y$$

We use the following constants, which we call *modal operators*.

$$\begin{aligned}
\dot{\neg} &: (\mathrm{IB})\mathrm{IB} & \dot{\neg} p &= \lambda x. \neg px \\
\dot{\wedge} &: (\mathrm{IB})(\mathrm{IB})\mathrm{IB} & p \mathbin{\dot{\wedge}} q &= \lambda x.\, px \wedge qx \\
\dot{\vee} &: (\mathrm{IB})(\mathrm{IB})\mathrm{IB} & p \mathbin{\dot{\vee}} q &= \lambda x.\, px \vee qx \\
\langle _ \rangle_n &: (\mathrm{IIB})(\mathrm{IB})\mathrm{IB} & \langle r \rangle_n p &= \lambda x. \exists Y.\, DY \wedge (\textstyle\bigwedge_{y \in Y} rxy \wedge py) \\
[_]_n &: (\mathrm{IIB})(\mathrm{IB})\mathrm{IB} & [r]_n p &= \lambda x. \forall Y.\, (\textstyle\bigwedge_{y \in Y} rxy) \wedge DY \rightarrow \textstyle\bigvee_{y \in Y} py \\
E_n &: (\mathrm{IB})\mathrm{IB} & E_n p &= \lambda x. \exists Y.\, DY \wedge \textstyle\bigwedge_{y \in Y} py \\
A_n &: (\mathrm{IB})\mathrm{IB} & A_n p &= \lambda x. \forall Y.\, DY \rightarrow \textstyle\bigvee_{y \in Y} py \\
\dot{_} &: \mathrm{IIB} & \dot{x} &= \lambda y. x \doteq y \\
T &: (\mathrm{IIB})\mathrm{B} & Tr &= \forall xyz. rxy \wedge ryz \rightarrow rxz
\end{aligned}$$

where $n \geq 0$ and $|Y| = n + 1$ in all equations

To the right of each constant is an equation defining its semantics. Formulas of the form $[r]_n tx$ are called *box formulas* or *boxes*, and formulas $\langle r\rangle_n tx$ are called *diamond formulas* or *diamonds*. The semantics of boxes and diamonds is defined following [3, 5]. Intuitively, it can be described as follows:

– $\langle r\rangle_n p$: There are at least $n+1$ r-successors satisfying p.

– $[r]_n p$: All r-successors but possibly n exceptions satisfy p.

Our language does not contain a dedicated symbol for the universal role. Instead, we use graded *global modalities* E_n and A_n, which are semantically equivalent to qualified number restrictions on the universal role. So, for instance, $E_1 p$ holds if there are at least two distinct states satisfying p. Formulas of the form Tr are called *transitivity assertions*. We assume the application of modal operators to have a higher precedence than regular functional application. So, for instance, we write $\dot\neg\langle r\rangle_2 \dot y \,\dot\vee\, p\; x$ for $((\dot\neg(\langle r\rangle_2(\dot y)))\,\dot\vee\, p)x$.

A *modal interpretation* $\mathfrak{M}$ is an interpretation of simple type theory that interprets B as the set $\{0,1\}$, $\bot$ as 0 (i.e., false), $\top$ as 1 (i.e., true), maps I to a non-empty set, gives the logical constants $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, $\forall$, $\doteq$ their usual meaning, and satisfies the equations defining the modal operators $\dot\neg$, $\dot\wedge$, $\dot\vee$, $\langle \_\rangle_n$, $[\_]_n$, E, A, $\dot{\_}$ and T. If $\mathfrak{M}t = 1$, we say that $\mathfrak{M}$ *satisfies* t. A formula is called *satisfiable* if it has a satisfying modal interpretation.

3 Branches

For the sake of simplicity, we will define our tableau calculus $\mathcal{T}$ on negation normal *modal expressions*, i.e., terms of the form:

$$t \;::=\; p \mid \dot\neg p \mid \dot x \mid \dot\neg\dot x \mid t \,\dot\wedge\, t \mid t \,\dot\vee\, t \mid \langle r\rangle_n t \mid [r]_n t \mid E_n t \mid A_n t$$

A *branch* Γ is a finite set of formulas s of the form

$$s \;::=\; tx \mid rxy \mid Tr \mid x\doteq y \mid x\not\doteq y \mid \bot \mid \alpha{:}[r]_n tx$$

where t is a negation normal modal expression. The new form $\alpha{:}[r]_n tx$ serves algorithmic purposes. The *label* α of such *label introductions* is taken from a countably infinite set of labels. Formulas of the form rxy are called *edges*. We use the formula $\bot$ to explicitly mark unsatisfiable branches. We call a branch Γ *closed* if $\bot \in \Gamma$. Otherwise, Γ is called *open*. An interpretation $\mathfrak{M}$ satisfies a branch Γ if $\mathfrak{M}$ satisfies all *proper* formulas on Γ, i.e., all formulas except for label introductions. Given a finite set of input formulas (i.e., a branch) Γ_0, our tableau calculus decides if Γ_0 is satisfiable. We call Γ_0 the *initial branch*. The initial branch must contain no edges or label introductions. This restriction is inessential for the expressiveness of the language since label introductions are semantically irrelevant, and edges rxy can equivalently be expressed as $\langle r\rangle_0 \dot y x$.

Let Γ be a branch. With $\sim_\Gamma$ we denote the least equivalence relation $\sim$ on nominals such that $x \sim y$ for every equation $x\doteq y \in \Gamma$. We define the *equational closure* $\tilde\Gamma$ of a branch Γ as

$$\begin{aligned}\tilde\Gamma \;:=\; &\Gamma \cup \{tx \mid t \text{ modal expression and } \exists x' :\; x' \sim_\Gamma x \text{ and } tx' \in \Gamma\}\\ &\cup \{rxy \mid \exists x', y' :\; x' \sim_\Gamma x \text{ and } y' \sim_\Gamma y \text{ and } rx'y' \in \Gamma\}\end{aligned}$$

4 Evidence and Pre-evidence

The proof of model existence for our calculus $\mathcal{T}$ proceeds in three stages. Applied to a satisfiable initial branch, the rules of $\mathcal{T}$ (defined in Sect. 5) construct a *quasi-evident* branch (defined in Sect. 6). We show that every quasi-evident branch can be extended to a *pre-evident* branch, which, in turn, can be extended to an *evident* branch. For evident branches, we show model existence.

We write $D_\Gamma X$ as an abbreviation for $\forall x, y \in X : \; x \neq y \implies x \dot{\neq} y \in \Gamma$. A branch Γ is called *evident* if it satisfies all of the following *evidence conditions*:

$$
\begin{aligned}
(t_1 \dot{\wedge} t_2)x \in \Gamma &\implies t_1 x \in \tilde{\Gamma} \text{ and } t_2 x \in \tilde{\Gamma} \\
(t_1 \dot{\vee} t_2)x \in \Gamma &\implies t_1 x \in \tilde{\Gamma} \text{ or } t_2 x \in \tilde{\Gamma} \\
\langle r \rangle_n t x \in \Gamma &\implies \exists Y\colon \; |Y| = n+1 \text{ and } D_\Gamma Y \text{ and } \{rxy, ty \mid y \in Y\} \subseteq \tilde{\Gamma} \\
[r]_n t x \in \Gamma &\implies |\{y \mid rxy \in \tilde{\Gamma}, \; ty \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \leq n \\
E_n t x \in \Gamma &\implies \exists Y\colon \; |Y| = n+1 \text{ and } D_\Gamma Y \text{ and } \{ty \mid y \in Y\} \subseteq \tilde{\Gamma} \\
A_n t x \in \Gamma &\implies |\{y \mid ty \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \leq n \\
\dot{x} y \in \Gamma &\implies x \sim_\Gamma y \\
\dot{\neg} \dot{x} y \in \Gamma &\implies x \not\sim_\Gamma y \\
x \dot{\neq} y \in \Gamma &\implies x \not\sim_\Gamma y \\
\dot{\neg} p x \in \Gamma &\implies px \notin \tilde{\Gamma} \\
Tr \in \Gamma &\implies \forall x, y, z : \; rxy \in \tilde{\Gamma} \text{ and } ryz \in \tilde{\Gamma} \implies rxz \in \tilde{\Gamma}
\end{aligned}
$$

A formula s is called *evident on Γ* if Γ satisfies the right-hand side of the evidence condition corresponding to s. For instance, $(t_1 \dot{\wedge} t_2)x$ is evident on Γ if and only if $\{t_1 x, t_2 x\} \subseteq \tilde{\Gamma}$.

We will now show that evident branches are satisfiable. Given a term t, we write $\mathcal{N}t$ for the set of nominals that occur in t. The notation is extended to sets of terms in the natural way: $\mathcal{N}\Gamma := \bigcup \{\mathcal{N}t \mid t \in \Gamma\}$.

Given a branch Γ, we construct the interpretation $\mathfrak{M}^\Gamma$ by taking as the domain of S the nominals on Γ, and interpreting propositional variables and roles as the smallest sets that are consistent with the respective assertions on Γ. To satisfy the equality constraints on Γ, all nominals that are equivalent modulo $\sim_\Gamma$ are mapped to the same fixed representative.

Let Γ be a branch and let $x_0 \in \mathcal{N}\Gamma$. Let ρ be a function from finite sets of nominals to nominals such that $\rho X \in X$ whenever X is nonempty. We define the interpretation $\mathfrak{M}^\Gamma$ as follows:

$$
\begin{aligned}
&\mathfrak{M}^\Gamma \mathrm{S} := \mathcal{N}\Gamma \\
&\mathfrak{M}^\Gamma x := \text{if } x \in \mathcal{N}\Gamma \text{ then } \rho\{y \in \mathcal{N}\Gamma \mid y \sim_\Gamma x\} \text{ else } x_0 \\
&\mathfrak{M}^\Gamma p := \{x \in \mathcal{N}\Gamma \mid px \in \tilde{\Gamma}\} \\
&\mathfrak{M}^\Gamma r := \{(x, y) \in (\mathcal{N}\Gamma)^2 \mid rxy \in \tilde{\Gamma}\}
\end{aligned}
$$

Note that in the last two lines of the definition, we interpret the set notation as a convenient description for the respective characteristic functions.

Theorem 4.1 (Model Existence). *If Γ is an evident branch, then $\mathfrak{M}^\Gamma$ satisfies Γ.*

Proof. Let Γ be an evident branch. For every $s \in \Gamma$, we show that $\mathfrak{M}^\Gamma$ satisfies s by induction on s. The details are straightforward. □

To simplify the treatment of transitivity, we introduce the notion of pre-evidence. We define the relation $\rhd^r_\Gamma$ as the least relation such that:

$$\begin{aligned} rxy \in \tilde{\Gamma} &\implies x \rhd^r_\Gamma y \\ x \rhd^r_\Gamma y \text{ and } y \rhd^r_\Gamma z \text{ and } Tr \in \Gamma &\implies x \rhd^r_\Gamma z \end{aligned}$$

We write $x \unrhd^r_\Gamma y$ iff $x \sim_\Gamma y$ or $x \rhd^r_\Gamma y$.

The *pre-evidence conditions* are obtained from the evidence conditions by omitting the condition for transitivity assertions and replacing the condition for boxes as follows:

$$[r]_n tx \in \Gamma \implies |\{y \mid x \rhd^r_\Gamma y \text{ and } ty \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \leq n$$

Pre-evidence of individual formulas is defined analogously to the corresponding evidence condition. Note that for all formulas but boxes and transitivity assertions, the notions of evidence and pre-evidence coincide.

We now show that every pre-evident branch can be extended to an evident branch. Let the *evidence closure* $\hat{\Gamma}$ of a branch Γ be defined as $\Gamma \cup \{rxy \mid x \rhd^r_\Gamma y\}$.

Proposition 4.1. $rxy \in \hat{\Gamma} \iff rxy \in \tilde{\hat{\Gamma}} \iff x \rhd^r_\Gamma y$

Theorem 4.2 (Evidence Completion). *Γ pre-evident $\implies \hat{\Gamma}$ evident*

Proof. Since $\hat{\Gamma}$ differs from Γ only in that $\hat{\Gamma}$ may contain more edges, and Γ is pre-evident, $\hat{\Gamma}$ satisfies all of the evidence conditions but possibly the ones for boxes and transitivity assertions. The evidence condition for transitivity assertions holds in $\hat{\Gamma}$ by Proposition 4.1 since $\rhd^r_\Gamma$ is transitively closed for every r such that $Tr \in \Gamma$. The condition for boxes is immediate by Proposition 4.1. □

5 Tableau Rules

The tableau rules of our calculus $\mathcal{T}$ are defined in Fig. 1. In the rules, we write $\exists x \in X : \Gamma(x)$ for $\Gamma(x_1) \mid \ldots \mid \Gamma(x_n)$, where $X = \{x_1, \ldots, x_n\}$ and $\Gamma(x)$ is a set of formulas parametrized by x. In case $X = \emptyset$, the notation translates to $\bot$. Dually, we write $\forall x \in X : \Gamma(x)$ for $\Gamma(x_1), \ldots, \Gamma(x_n)$ ($X = \{x_1, \ldots, x_n\}$). If $X = \emptyset$, the notation stands for the empty set of formulas.

The side condition of $\mathcal{R}_\Diamond$ uses the notion of quasi-evidence, which we will introduce in Sect. 6. For now, assume the rule is formulated with the restriction "$\langle r \rangle_n tx$ not evident on Γ".

A box formula $[r]_n tx$ is *subsumed* on Γ if there is a nominal y and a label α such that $y \unrhd^r_\Gamma x$ and $\alpha{:}[r]_n ty \in \Gamma$. The rule $\mathcal{R}_T$ is constrained to be applicable

$$\mathcal{R}_{\dot\wedge}\ \frac{(s \mathbin{\dot\wedge} t)x}{sx,\ tx} \qquad\qquad \mathcal{R}_{\dot\vee}\ \frac{(s \mathbin{\dot\vee} t)x}{sx \ \mid\ tx}$$

$$\mathcal{R}_{\Diamond}\ \frac{\langle r\rangle_n tx}{\forall y \in Y\colon\ rxy,\ ty,\ \forall z \in Y,\ y \neq z\colon\ y \dot{\neq} z}\ \begin{array}{l} Y \text{ fresh},\ |Y| = n+1, \\ \langle r\rangle_n tx \text{ not quasi-evident on } \Gamma \end{array}$$

$$\mathcal{R}_{\Box}\ \frac{[r]_n tx}{\exists y,z \in Y,\ y \neq z\colon\ y \doteq z \ \mid\ \exists y \in Y\colon\ ty}\ Y \subseteq \{y \mid x \rhd^r_\Gamma y\},\ |Y| = |Y/_{\sim_\Gamma}| = n+1$$

$$\mathcal{R}_T\ \frac{Tr,\ rxy}{\alpha{:}[r]_n tx}\ \alpha \text{ fresh},\ [r]_n tx \in \tilde\Gamma,\ [r]_n tx \text{ not subsumed on } \Gamma$$

$$\mathcal{R}_E\ \frac{E_n tx}{\forall y \in Y\colon\ ty,\ \forall z \in Y,\ y \neq z\colon\ y \dot{\neq} z}\ Y \text{ fresh},\ |Y| = n+1,\ E_n tx \text{ not evident on } \Gamma$$

$$\mathcal{R}_A\ \frac{A_n tx}{\exists y,z \in Y,\ y \neq z\colon\ y \doteq z \ \mid\ \exists y \in Y\colon\ ty}\ Y \subseteq \mathcal{N}\Gamma,\ |Y| = |Y/_{\sim_\Gamma}| = n+1$$

$$\mathcal{R}_N\ \frac{\dot{x}y}{x \doteq y} \qquad \mathcal{R}_{\bar N}\ \frac{\dot\neg \dot{x}y}{x \dot{\neq} y} \qquad \mathcal{R}^{\perp}_{\dot\neg}\ \frac{\dot\neg px}{\perp}\ px \in \tilde\Gamma \qquad \mathcal{R}^{\perp}_{\dot\neq}\ \frac{x \dot{\neq} y}{\perp}\ x \sim_\Gamma y$$

Γ is the branch to which a rule is applied. "Y fresh" stands for $Y \cap \mathcal{N}\Gamma = \emptyset$. "$\alpha$ fresh" stands for $\nexists t, x\colon\ \alpha{:}tx \in \Gamma$

Fig. 1. Tableau rules for $\mathcal{T}$

only to boxes that are not subsumed on Γ. This ensures, in particular, that $\mathcal{R}_T$ is applied at most once to each individual box formula on the branch.

A branch Δ is called a *proper extension* of a branch Γ if $\Delta \supseteq \Gamma$ and $\tilde\Delta \supsetneq \tilde\Gamma$. Note that if Δ is a proper extension of Γ, then in particular it holds $\Delta \supsetneq \Gamma$. The converse does not hold: Let $\Gamma := \{\dot{x}y,\ x \doteq z,\ z \doteq y\}$ and $\Delta := \Gamma \cup \{x \doteq y\}$. Then $\Delta \supsetneq \Gamma$ but Δ is not a proper extension of Γ. We implicitly restrict the applicability of the tableau rules so that a rule $\mathcal{R}$ is only applicable to a formula $s \in \Gamma$ if all of the alternative branches $\Delta_1, \ldots, \Delta_n$ resulting from this application are proper extensions of Γ.

Proposition 5.1 (Soundness). *Let $\Delta_1, \ldots, \Delta_n$ be the branches obtained from a branch Γ by a rule of $\mathcal{T}$. Then Γ is satisfiable if and only if there is some $i \in \{1, \ldots, n\}$ such that Δ_i is satisfiable.*

6 Blocking Conditions and Quasi-evidence

The restrictions on the applicability of the tableau rules given by the pre-evidence conditions are not sufficient for termination. Consider $\Gamma_0 := \{A_0 \langle r\rangle_0 px\}$. An application of $\mathcal{R}_A$ to Γ_0 yields $\Gamma_1 := \Gamma_0 \cup \{\langle r\rangle_0 px\}$, which can be extended by $\mathcal{R}_\Diamond$ to $\Gamma_2 := \Gamma_1 \cup \{rxy,\, py\}$. Now $\mathcal{R}_A$ is applicable again and yields $\Gamma_3 := \Gamma_2 \cup \{\langle r\rangle_0 py\}$, which in turn can be extended by $\mathcal{R}_\Diamond$, and so ad infinitum.

To obtain a terminating calculus, we restrict the rule $\mathcal{R}_\Diamond$ by weakening the notion of pre-evidence for diamond formulas. The weaker notion, called quasi-evidence, is then used in the side condition of $\mathcal{R}_\Diamond$ in place of pre-evidence. Quasi-evidence must be weak enough to guarantee termination but strong enough to preserve completeness.

The *edge graph* of a branch Γ is a labelled graph with the nodes $\mathcal{N}\Gamma$ and edges $\{(x, y) \mid \exists r : \ rxy \in \Gamma\}$, where a node x is labelled with all expressions t such that $tx \in \Gamma$, and an edge (x, y) is labelled with all roles r such that $rxy \in \Gamma$. A branch can always be represented graphically through its edge graph.

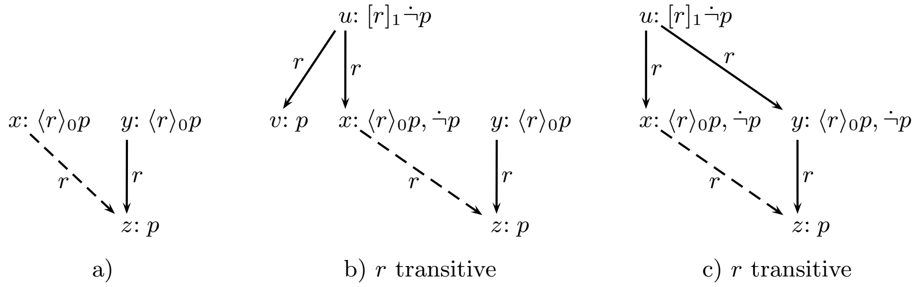

Fig. 2. Number restrictions and transitivity

In [14], the notion of quasi-evidence is based on the following observation. Let Γ be a branch and x, y be nominals such that: (1) x *has no* r*-successor on* Γ, i.e., there is no z such that $rxz \in \tilde{\Gamma}$, (2) for every r-diamond or r-box $tx \in \tilde{\Gamma}$, it holds $ty \in \tilde{\Gamma}$, and (3) all r-diamonds and r-boxes $sy \in \tilde{\Gamma}$ are evident on Γ. Then all r-diamonds and r-boxes $sx \in \tilde{\Gamma}$ can be made evident by extending Γ with $\{rxz \mid ryz \in \tilde{\Gamma}\}$. As an example, consider the edge graph in Fig. 2(a). There, the formula $\langle r\rangle_0 px$ can be made evident by adding the edge rxz (represented by the dashed arrow) to the branch. In the presence of transitivity, extending a branch Γ by an edge rxz may destroy the evidence of r-boxes tu such that $u \rhd^r_\Gamma x$ (Fig. 2(b)). Note, however, that adding an edge rxz cannot destroy the evidence of a box tu such that $u \rhd^r_\Gamma x$ if we already have $u \rhd^r_\Gamma z$ (Fig. 2(c)).

To deal with non-local constraints introduced by number restrictions on transitive roles, we refine the notion of a pattern and the quasi-evidence conditions from [14]. When blocking a nominal x we have to make sure not to violate any graded boxes at the predecessors of x. To track the relevant boxes we tag them with labels.

Given a role r, an r*-pattern* is a set consisting of modal expressions of the form μt, where $\mu \in \{\langle r\rangle_n, [r]_n \mid n \in \mathbb{N}\}$, and labels α, such that, for some n, t, x: $\alpha{:}[r]_n tx \in \Gamma$ (although not required by the definition, in all cases where patterns play a role for termination they will contain at least one diamond). We define:

$$x{:}_\Gamma\alpha \iff \exists r,n,t,y:\ \alpha{:}[r]_n ty \in \Gamma \text{ and } y \rhd^r_\Gamma x$$

We write $P^r_\Gamma x$ for the largest r-pattern P such that $P \subseteq \{\mu t \mid \mu tx \in \tilde{\Gamma}\} \cup \{\alpha \mid x{:}_\Gamma\alpha\}$. We call $P^r_\Gamma x$ the r-pattern of x on Γ. Looking back at Fig. 2 (b), we have $P^r_\Gamma x = \{\langle r\rangle_0 p\}$, $P^r_\Gamma u = \{[r]_1 \dot{\neg} p\}$, and $P^{r'}_\Gamma x = \emptyset$ for all $r' \neq r$. An r-pattern P is *expanded on* Γ if there are nominals x, y such that $rxy \in \Gamma$ and $P \subseteq P^r_\Gamma x$. In this case, we say that the nominal x *expands* P *on* Γ.

A diamond $\langle r\rangle_n sx \in \Gamma$ is *quasi-evident on* Γ if it is either evident on Γ or x has no r-successor on Γ and $P^r_\Gamma x$ is expanded on Γ. The rule $\mathcal{R}_\Diamond$ can only be applied to diamonds that are not quasi-evident. Note that whenever $\langle r\rangle_n sx \in \Gamma$ is quasi-evident but not evident (on Γ), there is a nominal y that expands $P^r_\Gamma x$.

The *quasi-evidence conditions* are obtained from the pre-evidence conditions by replacing the condition for diamond formulas and adding a condition for transitivity assertions and label introductions as follows:

$$\langle r\rangle_n tx \in \Gamma \implies \langle r\rangle_n tx \text{ is quasi-evident on } \Gamma$$
$$Tr \in \Gamma \implies \forall n,t,x\colon\ [r]_n tx \in \tilde{\Gamma} \implies \exists z,\alpha\colon\ z \unrhd^r_\Gamma x \text{ and } \alpha{:}[r]_n tz \in \Gamma$$
$$\alpha{:}[r]_n tx \in \Gamma \implies [r]_n tx \in \tilde{\Gamma} \text{ and } \exists y\colon rxy \in \Gamma \text{ and } \forall s,z\colon \alpha{:}sz \in \Gamma \implies s = [r]_n t$$

Proposition 6.1. *If Γ satisfies the quasi-evidence condition for label introductions and $\alpha{:}[r]_n tx \in \Gamma$, then for all y, $x \rhd^r_\Gamma y \iff y{:}_\Gamma\alpha$.*

Lemma 6.1. *Let Γ be a branch. Let $\{[r]_n tx, [r]_n ty\} \subseteq \tilde{\Gamma}$ such that $Tr \in \Gamma$ and $x \unrhd^r_\Gamma y$. Then:* $[r]_n tx$ is pre-evident on $\Gamma \implies [r]_n ty$ is pre-evident on Γ.

Proof. Let Γ be a branch such that $\{[r]_n tx, [r]_n ty\} \subseteq \tilde{\Gamma}$, $Tr \in \Gamma$ and $x \unrhd^r_\Gamma y$. Because $\rhd^r_\Gamma$ is transitively closed, we have $x \rhd^r_\Gamma z$ whenever $y \rhd^r_\Gamma z$. The claim follows. ◻

Lemma 6.2. *Let Γ be a quasi-evident branch. Let $\langle r\rangle_n sx \in \Gamma$ be not evident on Γ, y be a nominal that expands $P^r_\Gamma x$ on Γ, and $\Delta := \Gamma \cup \{rxz \mid ryz \in \tilde{\Gamma}\}$. Then:*

1. *$\forall z:\ rxz \in \tilde{\Delta} \iff ryz \in \tilde{\Gamma}$ and $x \rhd^r_\Delta z \iff y \rhd^r_\Gamma z$,*
2. *$\forall m,t:\ \langle r\rangle_m t \in P^r_\Gamma x \implies \langle r\rangle_m tx$ is evident on Δ,*
3. *$\langle r\rangle_n sx$ is evident on Δ,*
4. *$\forall r',m,t,z:\ \langle r'\rangle_m tz$ is evident on $\Gamma \implies \langle r'\rangle_m tz$ is evident on Δ,*
5. *Δ is quasi-evident.*

Proof. We begin with (1). Let z be a nominal. We only show $rxz \in \tilde{\Delta} \Leftrightarrow ryz \in \tilde{\Gamma}$. The other claim follows by induction on the construction of $\rhd^r_\Gamma$ and $\rhd^r_\Delta$. By construction, it holds $ryz \in \tilde{\Gamma} \Rightarrow rxz \in \Delta$. The converse implication holds by the fact that $\langle r\rangle_n sx$ is quasi-evident but not evident on Γ, meaning that x has no r-successor on Γ. It remains to show: $rxz \in \Delta \Leftrightarrow rxz \in \tilde{\Delta}$. The direction from left to right is obvious. For the other direction, assume $rxz \in \tilde{\Delta}$. Then there are x', z' such that $x' \sim_\Gamma x$, $z' \sim_\Gamma z$, and $rx'z' \in \Delta$. Since x has no r-successor on Γ, neither does x'. Hence, since $rx'z' \in \Delta - \Gamma$, we must have $x' = x$, and so $rxz' \in \Delta$. But then $ryz' \in \tilde{\Gamma}$, and consequently, $ryz \in \tilde{\Gamma}$. The claim follows by the definition of Δ.

Now to (2). Let $\langle r\rangle_m t \in P^r_\Gamma x$. Since $P^r_\Gamma y \supseteq P^r_\Gamma x$, in particular it holds $\langle r\rangle_m ty \in \tilde{\Gamma}$, i.e., there is some $y' \sim_\Gamma y$ such that $\langle r\rangle_m ty' \in \Gamma$. By (1), it suffices to show that $\langle r\rangle_m ty$ is evident on Γ. This is the case since $\langle r\rangle_m ty'$ is quasi-evident on Γ (as Γ is quasi-evident) and y' has an r-successor on Γ (as y has one on Γ).

Claim (3) immediately follows from (2), and (4) is obvious as the evidence of diamonds on a branch cannot be destroyed by adding edges.

Now to (5). Note that the quasi-evidence condition for transitivity assertions holds in Δ as $\trianglerighteq^r_\Gamma \subseteq \trianglerighteq^r_\Delta$. The quasi-evidence of diamonds $\langle r\rangle_m tx \in \Delta$ holds by (2). So, the only conditions that might in principle be violated in Δ are:
a) the pre-evidence condition for boxes $[r]_m tx \in \tilde{\Delta}$ and
b) the pre-evidence condition for boxes $[r]_m tz \in \Delta$ such that $z \triangleright^r_\Delta x$, if $Tr \in \Gamma$.
For (a), it holds $[r]_m ty \in \tilde{\Gamma}$ as $P^r_\Gamma y \supseteq P^r_\Gamma x = P^r_\Delta x$. Hence by (1) it suffices to show that $[r]_m ty$ is pre-evident on Γ, which is the case since Γ is quasi-evident.
For (b), by the quasi-evidence condition for transitivity assertions, there is a nominal u and a label α such that $u \trianglerighteq^r_\Gamma z$ and $\alpha{:}[r]_m tu \in \Gamma$. Since $Tr \in \Gamma$, $u \trianglerighteq^r_\Gamma z$ and $z \triangleright^r_\Delta x$, it holds $u \triangleright^r_\Gamma x$. Then $x{:}_\Gamma\alpha$ and, by the quasi-evidence condition for label introductions, $[r]_m tu \in \tilde{\Gamma}$. By Lemma 6.1, it suffices to show that $[r]_m tu$ is pre-evident on Δ. Since $P^r_\Gamma y \supseteq P^r_\Gamma x$, we have $y{:}_\Gamma\alpha$ and hence $u \triangleright^r_\Gamma y$ (Proposition 6.1). So, by (1), $x \triangleright^r_\Delta v$ implies $u \triangleright^r_\Gamma v$ for all nominals v, and consequently, $\forall v:\ u \triangleright^r_\Delta v \Leftrightarrow u \triangleright^r_\Gamma v$. The claim follows since $[r]_m tu$ is pre-evident on Γ. ☐

For an illustration of Lemma 6.2, let the edge graph in Fig. 2(a) (without the dashed arrow) represent Γ. Then $\langle r\rangle_0 px$ is quasi-evident but not evident on Γ, and y expands $P^r_\Gamma x$. The graph with the dashed arrow added corresponds to the branch Δ in the lemma. The five claims for Γ and Δ are easy to verify.

Theorem 6.1 (Pre-evidence Completion). *For every quasi-evident branch Γ there is a pre-evident branch Δ such that $\Gamma \subseteq \Delta$.*

Proof. For every branch Γ, we define: $\varphi\Gamma := |\{\langle r\rangle_n sx \mid \langle r\rangle_n sx \in \Gamma$ and $\langle r\rangle_n sx$ is not evident on $\Gamma\}|$. Let Γ be quasi-evident. We proceed by induction on $\varphi\Gamma$. If $\varphi\Gamma = 0$, then Γ is pre-evident and we are done. Otherwise, there is a diamond $\langle r\rangle_n sx \in \Gamma$ that is not pre-evident on Γ. Let y be a nominal that expands $P^r_\Gamma x$ on Γ, and let $\Gamma' := \Gamma \cup \{rxz \mid ryz \in \tilde{\Gamma}\}$. By Lemma 6.2(3-5), Γ' is quasi-evident and $\varphi\Gamma' < \varphi\Gamma$. So, by the inductive hypothesis, there is some pre-evident branch Δ such that $\Delta \supseteq \Gamma' \supseteq \Gamma$. ☐

We write $\Gamma \xrightarrow{\mathcal{R}} \Delta$ to denote that Δ is obtained from Γ by a single application of the rule $\mathcal{R}$. We write $\Gamma \to \Delta$ if there is some $\mathcal{R}$ such that $\Gamma \xrightarrow{\mathcal{R}} \Delta$. A branch is called *maximal* if it cannot be extended by any tableau rule.

Lemma 6.3. *Let Γ be a branch that is obtained from an initial branch. Then Γ satisfies the quasi-evidence condition for label introductions.*

Proof. Let $\Gamma_0 \to \ldots \to \Gamma_n$ be a derivation such that Γ_0 is an initial branch and $\Gamma_n = \Gamma$. The claim is shown by induction on n. Note that the claim is trivial for $n = 0$ since initial branches must contain no edges or label introductions. □

In conjunction with Theorems 4.1, 4.2 and 6.1, the following theorem shows that open maximal branches are satisfiable. Taken together with the termination argument in Section 7, this establishes the completeness of our calculus.

Theorem 6.2 (Quasi-evidence). *Every open and maximal branch obtained in $\mathcal{T}$ from an initial branch is quasi-evident.*

Proof. Let Γ be an open and maximal branch obtained from an initial branch. We show that every $s \in \Gamma$ that is not of the form px, rxy or $x\doteq y$ is either pre-evident or quasi-evident on Γ by induction on the size of s. Quasi-evidence for label introductions follows by Lemma 6.3. □

7 Termination

We will now show that every tableau derivation is finite. Since the tableau rules are all finitely branching, by König's lemma it suffices to show that the construction of every individual branch terminates. Since rule application always produces proper extensions of branches, it then suffices to show that the size (i.e., cardinality) of an individual branch is bounded. First, we show that the size of a branch Γ is bounded by a function in the number of nominals on Γ. Then, we show that this number itself is bounded, completing the termination proof.

We write $\mathcal{S}\Gamma$ for the set of all modal expressions occurring on Γ, possibly as subterms of other expressions, and $\mathrm{Rel}\,\Gamma$ for the set of all roles that occur on Γ. Crucial for the termination argument is the fact the tableau rules cannot introduce any modal expressions that do not already occur on the initial branch.

Proposition 7.1. *If Γ, Δ are branches such that Δ is obtained from Γ by any rule of $\mathcal{T}$, then $\mathcal{S}\Delta = \mathcal{S}\Gamma$.*

For every pair of nominals x, y and every role r, a branch Γ may contain an edge rxy, an equation $x\doteq y$ or a disequation $x\not\doteq y$. For every expression $s \in \mathcal{S}\Gamma$, Γ may contain a formula sx. The tableau rules can introduce at most one formula $\alpha{:}[r]_n tx$ for each box expression $[r]_n t$ and each nominal x. Finally, a branch may contain $\bot$. So, since the initial branch Γ_0 contains no formulas of the form $\alpha{:}tx$, the size of Γ derived from Γ_0 is bounded by $|\mathrm{Rel}\,\Gamma| \cdot |\mathcal{N}\Gamma|^2 + 2|\mathcal{N}\Gamma|^2 + 2|\mathcal{S}\Gamma| \cdot |\mathcal{N}\Gamma| + 1$. By Proposition 7.1, we know that $|\mathcal{S}\Gamma|$ and $|\mathrm{Rel}\,\Gamma|$ depend only on Γ_0.

By the above, it suffices to show that $|\mathcal{N}\Gamma|$ is bounded in the sum of the sizes of the input formulas (of which there are only finitely many). We do so by giving a bound on the number of applications of $\mathcal{R}_\Diamond$ and $\mathcal{R}_E$ that can occur in the derivation of a branch, which suffices since the two rules are the only ones that can introduce new nominals.

For $\mathcal{R}_E$, we do so by defining $\psi_E\Gamma := \{E_n s \in \mathcal{S}\Gamma \mid \exists x \in \mathcal{N}\Gamma :\ E_n sx \text{ is not evident on } \Gamma\}$ and showing that $|\psi_E\Gamma|$ decreases with every application of $\mathcal{R}_E$ (and is non-increasing otherwise, which is obvious).

Proposition 7.2. $\Gamma \overset{\mathcal{R}_E}{\to} \Delta \implies |\psi_E \Gamma| > |\psi_E \Delta|$

The proof proceeds analogously to the corresponding arguments in [15, 16].

Now we show that $\mathcal{R}_\Diamond$ can be applied only finitely often. Since $\mathrm{Rel}\,\Gamma$ is bounded, it suffices to show that $\mathcal{R}_\Diamond$ can be applied only finitely often for each role. Since $\mathcal{R}_\Diamond$ is only applicable to diamonds that are not quasi-evident, we have:

Proposition 7.3. *If $\mathcal{R}_\Diamond$ is applicable to a formula $\langle r\rangle_n sx \in \Gamma$, then either*

1. *x has an r-successor on Γ, or*
2. *$P^r_\Gamma x$ is not expanded on Γ.*

Since $\Gamma \to \Delta$ implies $\tilde{\Gamma} \subseteq \tilde{\Delta}$, it holds:

Proposition 7.4. *Let $s \in \Gamma$ be a diamond formula and $\Gamma \to \Delta$.*

1. *If s is evident on Γ, then s is evident on Δ.*
2. *If Δ is obtained from Γ by applying $\mathcal{R}_\Diamond$ to s, then s is evident on Δ.*

Proposition 7.5. *Let $\Gamma \to \Delta$, $x \in \mathcal{N}\Gamma$, and P be an r-pattern.*

1. *$P^r_\Gamma x \subseteq P^r_\Delta x$.*
2. *If P is expanded on Γ, then P is expanded on Δ.*

In the case of [14], the bound on the number of applications of $\mathcal{R}_\Diamond$ for each role r can be given as $|\mathrm{Pat}^r \Gamma_0|$ where Γ_0 is the initial branch and $\mathrm{Pat}^r \Gamma := \mathcal{P}(\{\langle r\rangle_n s \mid \langle r\rangle_n s \in \mathcal{S}\Gamma\} \cup \{[r]_n s \mid [r]_n s \in \mathcal{S}\Gamma\})$. The present situation is more complex since now patterns may contain labels in addition to modal expressions. Unlike $\mathcal{S}\Gamma$, the set of labels on the branch may grow during tableau construction. Still, we can bound the number of applications of $\mathcal{R}_\Diamond$ for every given set of labels.

A rule $\mathcal{R}$ is said to be applied to a nominal $x \in \mathcal{N}\Gamma$ if $\mathcal{R}$ is applied to a formula $tx \in \Gamma$. Given a pattern P, we define $\mathcal{A}P := \{\alpha \mid \alpha \in P\}$. Let $N^{\Gamma_0}_{\langle r\rangle}$ be the number of distinct r-diamonds occurring on Γ_0: $N^{\Gamma_0}_{\langle r\rangle} := |\{\langle r\rangle_k t \mid \langle r\rangle_k t \in \mathcal{S}\Gamma_0\}|$. Let Δ be obtained from Γ by applying $\mathcal{R}_\Diamond$ to a formula $\langle r\rangle_n sx \in \Gamma$ such that $P^r_\Gamma x$ is not expanded on Γ. Clearly, $P^r_\Delta x$ must be expanded on Δ. Hence, let us call such an application of $\mathcal{R}_\Diamond$ *pattern-expanding.*

Lemma 7.1. *Let Γ_0 be an initial branch and $\Gamma_0 \to \Gamma_1 \to \dots$ a derivation. Let r be a role, A a set of labels, and*

$$I^r_A := \{i \mid \exists x : \Gamma_{i+1} \text{ is obtained from } \Gamma_i \text{ by applying } \mathcal{R}_\Diamond \text{ to } x \text{ and } \mathcal{A}(P^r_{\Gamma_i} x) = A\}$$

Then $|I^r_A| \le 2^{|A|} \cdot |\mathrm{Pat}^r \Gamma_0| \cdot N^{\Gamma_0}_{\langle r\rangle}$.

Proof. Let $\Gamma_0 \to \Gamma_1 \to \dots$ be a derivation, r a role and A a set of labels. We begin with two observations:

1. For every set B of labels, there are at most $|\mathrm{Pat}^r \Gamma_0|$ distinct patterns P such that $\mathcal{A}P = B$. Hence, by Proposition 7.5 (2), for every B there are at most $|\mathrm{Pat}^r \Gamma_0|$ pattern-expanding applications of $\mathcal{R}_\Diamond$ in the entire derivation,

i.e., at most $|\mathrm{Pat}^r \Gamma_0|$ indices $i \in I_B^r$ such that the application of $\mathcal{R}_\Diamond$ to Γ_i is pattern-expanding. Let us denote the set of such indices by J_B^r.

2. By Propositions 7.4 and 7.5 (2), every pattern-expanding application of $\mathcal{R}_\Diamond$ to a nominal x is followed by at most $N_{\langle r\rangle}^{\Gamma_0} - 1$ applications of $\mathcal{R}_\Diamond$ to nominals that are equivalent to x at the time of the respective application (clearly, none of these following applications is pattern-expanding).

By definition, every index in I_A^r corresponds to an application of $\mathcal{R}_\Diamond$. Let $i \in I_A^r$ and let x be the nominal to which $\mathcal{R}_\Diamond$ is applied on Γ_i. By Proposition 7.3, either the application is pattern-expanding or x already has a successor on Γ_i. In the latter case, the application must be preceded by a pattern-expanding application of $\mathcal{R}_\Diamond$ to some nominal y that is equivalent to x ($x \sim_{\Gamma_i} y$). As for the index j corresponding to this preceding application, by Proposition 7.5 (1), we must have $j \in J_B^r$ for some $B \subseteq A$. By the above two observations, we obtain:

$$
\begin{aligned}
|I_A^r| &\leq |J_A^r| + \sum_{B \subseteq A} |J_B^r| \cdot (N_{\langle r\rangle}^{\Gamma_0} - 1) \\
&\leq |\mathrm{Pat}^r \Gamma_0| + 2^{|A|} \cdot |\mathrm{Pat}^r \Gamma_0| \cdot (N_{\langle r\rangle}^{\Gamma_0} - 1) \leq 2^{|A|} \cdot |\mathrm{Pat}^r \Gamma_0| \cdot N_{\langle r\rangle}^{\Gamma_0} \qquad \square
\end{aligned}
$$

A set of labels A is called a *pattern space* for a role r on a branch Γ if there is some $x \in \mathcal{N}\Gamma$ such that $\mathcal{A}(P_\Gamma^r x) = A$. By Lemma 7.1, it suffices to show that for each role r, the number of pattern spaces created in a derivation is bounded.

Lemma 7.2. *Let Γ_0 be an initial branch, r a role and A a set of labels. There is a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every derivation $\Gamma_0 \to \Gamma_1 \to \ldots$:*

$$|\{x \mid \exists i, y:\ i \geq 0 \text{ and } \mathcal{A}(P_{\Gamma_i}^r x) = A \text{ and } rxy \in \Gamma_i\}| \leq f(|A|)$$

Proof. Let r and $\Gamma_0 \to \Gamma_1 \to \ldots$ be as required. Let $X_A := \{x \mid \exists i, y:\ i \geq 0$ and $\mathcal{A}(P_{\Gamma_i}^r x) = A$ and $rxy \in \Gamma_i\}$. We proceed by induction on $n := |A|$. For every $x \in X_A$, let i_x be the least i such that

1. $\mathcal{A}(P_{\Gamma_i}^r x) = A$, and
2. for some y, $rxy \in \Gamma_i$.

Since Γ_0 is an initial branch, it contains no edges, and so $i_x \geq 1$. No single rule application can make 1 and 2 true at the same time. Hence, for every $x \in X_A$ exactly one of the following is true:

Case $\mathcal{A}(P_{\Gamma_{i_x-1}}^r x) \subsetneq A$. Then there is some y such that $rxy \in \Gamma_{i_x-1}$. So, $x \in X_B$ for some proper subset B of A. Clearly, this case is only possible if $|A| > 0$.

Case $\nexists y:\ rxy \in \Gamma_{i_x-1}$. Then $\mathcal{A}(P_{\Gamma_{i_x-1}}^r x) = A$. So, $i_x - 1$ belongs to the set I_A^r from Lemma 7.1. This is the only case possible if $|A| = 0$.

By the above, f can be defined as follows:

$$
\begin{aligned}
f0 &:= |\mathrm{Pat}^r \Gamma_0| \cdot N_{\langle r\rangle}^{\Gamma_0} \\
fn &:= 2^n \cdot |\mathrm{Pat}^r \Gamma_0| \cdot N_{\langle r\rangle}^{\Gamma_0} + \sum_{k=0}^{n-1} \binom{n}{k} \cdot fk \qquad \text{if } n > 0 \qquad \square
\end{aligned}
$$

We define the *level* of an r-pattern P on Γ as:

$$L_\Gamma P := |\{[r]_m t \in \mathcal{S}\Gamma \mid \exists \alpha, y : \ \alpha \in P \text{ and } \alpha{:}[r]_m t y \in \Gamma\}|$$

A label α is said to be *generated at level* n in a derivation $\Gamma_0 \to \Gamma_1 \to \dots$ if there is some $i \geq 0$ such that α is generated by an application of $\mathcal{R}_T$ extending Γ_i by a formula $\alpha{:}[r]_m t x$, and $L_{\Gamma_i}(P^r_{\Gamma_i} x) = n$.

Lemma 7.3. *Let $\Gamma_0 \to \Gamma_1 \to \dots$ be a derivation where Γ_0 is initial and $Tr \in \Gamma_0$. Let $x \in \mathcal{N}\Gamma_i$. Then every label $\alpha \in P^r_{\Gamma_i} x$ is generated at level strictly less than $L_{\Gamma_i}(P^r_{\Gamma_i} x)$.*

Proof. Assume, by contradiction, Γ_i, r, and x are all as required and there is some $\alpha \in P^r_{\Gamma_i} x$ such that α is generated at level $m \geq L_{\Gamma_i}(P^r_{\Gamma_i} x)$. Then there is some $j < i$ such that α is generated by an application of $\mathcal{R}_T$ to some $ryz \in \Gamma_j$ such that $y \rhd^r_{\Gamma_i} x$ and $L_{\Gamma_j}(P^r_{\Gamma_j} y) = m$. Then $\mathcal{A}(P^r_{\Gamma_j} y) \cup \{\alpha\} \subseteq \mathcal{A}(P^r_{\Gamma_k} x')$ and hence (by the applicability restriction on $\mathcal{R}_T$) $L_{\Gamma_k}(P^r_{\Gamma_k} x') > m$ holds for all $k \geq j+1$ and all x' such that $y \rhd^r_{\Gamma_k} x'$. Consequently, $L_{\Gamma_i}(P^r_{\Gamma_i} x) > m \geq L_{\Gamma_i}(P^r_{\Gamma_i} x)$. Contradiction ☐

By Lemma 7.3, the number of pattern spaces with level n (i.e., pattern spaces whose patterns have level n) is bounded from above by 2^m, where m is the number of labels generated at levels less than n. Clearly, the level of r-patterns in a derivation from Γ_0 is bounded by the number $N^{\Gamma_0}_{[r]}$ of distinct r-boxes occurring on Γ_0 ($N^{\Gamma_0}_{[r]} := |\{[r]_k t \mid [r]_k t \in \mathcal{S}\Gamma_0\}|$). Also, by the applicability restriction on $\mathcal{R}_T$ (non-subsumption), no labels can be generated at level $N^{\Gamma_0}_{[r]}$. Hence, in order to show that the number of pattern spaces created during a derivation is bounded, it suffices to bound the number of labels generated at all levels less than $N^{\Gamma_0}_{[r]}$. A label α is called *r-label* (in a derivation $\Gamma_0 \to \Gamma_1 \to \dots$) if there are i, n, t, x such that $\alpha{:}[r]_n t x \in \Gamma_i$.

Lemma 7.4. *Let Γ_0 be an initial branch and $Tr \in \Gamma_0$. There is a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every derivation $\Gamma_0 \to \Gamma_1 \to \dots$ and $0 \leq n < N^{\Gamma_0}_{[r]}$: $|\{\alpha \mid \exists m < n : \ \alpha \text{ is an } r\text{-label generated at level } m\}| \leq fn$.*

Proof. We define f by induction on n. Let $A_m := \{\alpha \mid \exists k < m : \ \alpha \text{ is an } r\text{-label generated at level } k\}$. Clearly, $A_0 = \emptyset$. A new label can only be generated by an application of $\mathcal{R}_T$. Therefore, by the applicability condition of $\mathcal{R}_T$:

$$|A_n| \leq N^{\Gamma_0}_{[r]} \cdot |\{x \mid \exists i, y : \ i \geq 0 \text{ and } L_{\Gamma_i}(P^r_{\Gamma_i} x) \leq n-1 \text{ and } rxy \in \Gamma_i\}|$$

By Lemma 7.3, for all $n > 0$:

$$|A_n| \leq N^{\Gamma_0}_{[r]} \cdot |\bigcup_{B \subseteq A_{n-1}} \{x \mid \exists i, y : \ i \geq 0 \text{ and } \mathcal{A}(P^r_{\Gamma_i} x) = B \text{ and } rxy \in \Gamma_i\}|$$

Then, by Lemma 7.2, there is a function g such that, for all $n > 0$:

$$|A_n| \leq N^{\Gamma_0}_{[r]} \cdot \sum_{k=0}^{|A_{n-1}|} \binom{|A_{n-1}|}{k} \cdot gk \leq N^{\Gamma_0}_{[r]} \cdot 2^{|A_{n-1}|} \cdot g(|A_{n-1}|)$$

Hence, we can define $f0 := 0$ and, for $n > 0$, $fn := N^{\Gamma_0}_{[r]} \cdot 2^{f(n-1)} \cdot g(f(n-1))$ □

By Lemma 7.1, for every role r the number of applications of $\mathcal{R}_\Diamond$ is bounded by $\sum_{A \in \Phi} 2^{|A|} \cdot |\mathrm{Pat}^r\, \Gamma_0| \cdot N^{\Gamma_0}_{\langle r \rangle}$ where $\Phi := \{A \mid \exists i \geq 0\colon\ A$ is a pattern space for r on $\Gamma_i\}$. Using Lemma 7.3, this bound can be approximated from above by $|\mathrm{Pat}^r\, \Gamma_0| \cdot N^{\Gamma_0}_{\langle r \rangle} \cdot N^{\Gamma_0}_{[r]} \cdot (2^{2f(N^{\Gamma_0}_{[r]})})$ where f is the function from Lemma 7.4. Since we have only finitely many roles, together with Proposition 7.2, this gives us a bound on $|\mathcal{N}\Gamma|$ that we need for termination. Since f is clearly non-elementary in its argument, the bound is non-elementary.

8 Conclusion

To account for non-local constraints introduced by number restrictions on transitive roles, the notion of patterns from [14] needs to be extended. The extension is semantically intuitive and allows for a simple proof of model existence. As it comes to termination, the reasoning in [14] needs to be refined considerably.

The termination proof establishes a non-elementary complexity bound for the associated decision procedure. Presently, we do not know if this bound is tight. The NExpTime completeness result for (nominal-free) graded modal logic over transitive frames by Kazakov and Pratt-Hartmann [18] gives us a lower bound for the complexity of $\mathcal{SOQ}^+$ and hence of the decision procedure ([19] provides no complexity bounds). Despite the potentially high worst-case complexity of our procedure, we believe it to be well-suited for efficient implementation. In fact, on problems that do not contain number restrictions on transitive roles, the complexity of the procedure matches the NExpTime bound of [14], which is even lower than the 2-NExpTime bound established for practically successful procedures of [8, 13, 12, 9, 10].

Schröder and Pattinson [24] show concept satisfiability decidable in the presence of role hierarchies and number restrictions on transitive roles, provided the semantics is restricted to tree-like roles. They argue that the resulting logic, $\mathcal{PHQ}$, may be better suited for modeling parthood relations than the established logics extending $\mathcal{SH}$. We believe that our current approach for $\mathcal{SOQ}^+$ may be adapted to obtain an efficient tableau calculus for $\mathcal{PHQ}$.

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