Collisions and their Catenations: Ultimately Periodic Tilings of the Plane

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Abstract. Motivated by the study of cellular automata algorithmic and dynamics, we investigate an extension of ultimately periodic words to two-dimensional infinite words: collisions. A natural composition operation on tilings leads to a catenation operation on collisions. By existence of aperiodic tile sets, ultimately periodic tilings of the plane cannot generate all possible tilings but exhibit some useful properties of their one-dimensional counterparts: ultimately periodic tilings are recursive, very regular, and tiling constraints are easy to preserve by catenation. We show that, for a given catenation scheme of finitely many collisions, the generated set of collisions is semi-linear.

1 Introduction

The theory of regular languages, sets of one-dimensional sequences of letters sharing some regularities, has been well studied since the fifties. Finite state machines [18], regular languages [14, 5], computing devices with bounded memory, monadic second-order logic [4]: various point of views lead to a same robust notion of regular languages. The concept extends to infinite words and various other one-dimensional structures. Unfortunately, when considering two-dimensional words – partial mappings from the plane $\mathbb{Z}^2$ to a finite alphabet – such a robust common object fails to emerge: automata on the plane, picture languages, second-order logic, all lead to different notions of regular languages [9]. A first difficulty arises from the definition of a finite word: should it be any partial mapping with a finite support? Should it be rectangles filled with letters? Should it be any mapping with a connected support for some particular connexity notion? A second difficulty arises from the complexity of two-dimensional patterns: in the simplest case of uniform local constraints, *i.e. tilings*, knowing whether a given finite pattern is a factor of a valid tiling (of the whole plane) is already undecidable [1].

In the present paper, we investigate a particular family of recursive tilings of the plane endowed with a catenation operation. Our definition of an ultimately periodic tiling, a *collision*, is inspired by geometrical considerations on one-dimensional cellular automata space-time diagrams and tilings. It can be thought of as an extension of the notion of ultimately periodic bi-infinite words to two-dimensional words. These objects provide a convenient tool to describe synchronization problems in cellular automata algorithmic.

One-dimensional cellular automata [13] are dynamical systems whose configurations consist of bi-infinite words on a given finite alphabet. The system evolves by applying uniformly and synchronously a locally defined transition rule. The value at each position, or *cell*, of a configuration only depends on the values of the cells on its neighborhood at the previous time step. To discuss the dynamics or to describe algorithmic constructions, it is often convenient to consider space-time diagrams rather than configurations. A space-time diagram is a drawing of a particular orbit of the system: configurations are depicted one on top of the other, from bottom to top, by successively applying the transition rule, as depicted on Fig. 1. This representation permits to draw away the time-line and discuss the structure of emerging two-dimensional patterns. Formally, this is equivalent to consider tilings of half the plane with a special kind of local constraint, oriented by the time-line.

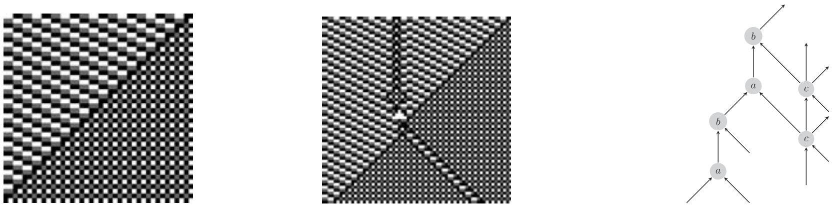
Time goes from bottom to top. Each letter is represented by a different color.

Fig. 1 Space-time diagram of a one-dimensional cellular automaton

Let us give first an informal overview of what collisions are and where they come from. An ultimately periodic configuration consists of two infinite periodic words separated by a finite non-periodic word. As transitions of cellular automata are locally defined, the image of an ultimately periodic configuration is an ultimately periodic configuration such that: for each periodic part, the period in the image divides the period in the preimage; for the non-periodic part, it can only grow by a finite size depending on the local rule. If, by iterating the transition rule of the cellular automaton, the size of the non-periodic part of the configurations remains bounded, then the orbit of the ultimately periodic configuration is, up to a translation, ultimately periodic. When considering this ultimately periodic behavior from the space-time diagram point of view, one can see some kind of *particle*: a localized structure moving with a rational slope in a periodic *background* environment, as depicted on Fig. 2a.

As particles are ultimately periodic configurations, one can construct more complicated configurations by putting particles side by side, ensuring that the non-periodic parts are far enough from each other, and that the periodic parts of two particles put side by side are the same and well aligned. If the non-periodic part of several particles (two or more) becomes near enough in the orbit, complex interactions might occur. If the interaction is localized in both space and time, as depicted on Fig. 2b, this interaction is called a *collision*.

Particles and collisions provide a convenient tool in the study of cellular automata. When constructing two-dimensional cellular automata, like in historical



(a) particle (b) collision (c) network of collisions

Fig. 2 Particles and collisions generated by ultimately periodic configurations

constructions of von Neumann [20] and Codd [6], particles are a convenient way to convey quanta of information from place to place. The most well known example of particle is certainly the glider of the Game of Life used by Conway et al. to embed computation inside the Game of Life [2] by using particular behavior of glider collisions. When using one-dimensional cellular automata to recognize languages or to compute functions, a classical tool is the notion of signal introduced by Fischer [8] and later developed by Mazoyer and Terrier [16, 17]: signals and their interactions are simple kinds of particles and collisions. Particles appear even in the classification of cellular automata dynamics: in its classification [21], Wolfram identifies what he calls class 4 cellular automata where *"(...) localized structures are produced which on their own are fairly simple, but these structures move around and interact with each other in very complicated ways. (...)"* A first study of particles interaction was proposed by Boccara et al. [3], latter followed by Crutchfield et al. [12]: these works focus on particles and bounding the number of possible collisions they can produce. Finally, the proof by Cook of the universality of rule 110 [7] is a typical construction involving a huge number of particles and collisions: once the gadgets and the simulation are described, the main part of the proof consists of proving that particles are well synchronized and that collisions occur exactly as described in the simulation.

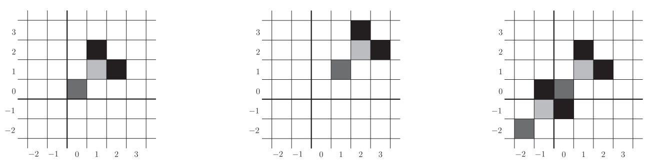
When dealing with space-time diagrams consisting of only particles and collisions, a second object is often used: a planar map describing the collisions and their interactions. When identifying particles and collisions in space-time diagrams, in the style of Boccara et al. [3], one builds the planar map to give a compact description of the diagram, as depicted on Fig. 2(c). When describing algorithmic computation, in the style of Fischer [8], one describes a family of planar maps as a scheme of the produced space-time diagrams.

The aim of the present paper is to define particles and collisions, describe how collisions can be catenated, introduce collisions schemes as planar maps and discuss the construction of finite catenations from collisions schemes. All the necessary material is defined in section 2 followed by basic catenation of tilings in section 3. Collisions and their catenations are formally introduced in section 4. The main result on catenation is presented in section 5.

2 Definitions

In the remaining of this paper, every discussion occurs in the *two-dimensional plane* $\mathbb{Z}^2$ partially colored with the letters of a given finite alphabet Σ. A *pattern* is a subset of $\mathbb{Z}^2$. A *cell* c of a given pattern P is an element $c \in P$. A *vector* is an element of the group $\left(\mathbb{Z}^2, +\right)$ of translations in the plane. A *coloring* $\mathcal{C}$ is a partial map from $\mathbb{Z}^2$ to Σ. The *support* of a *coloring* $\mathcal{C}$ is denoted by $\mathrm{Sup}(\mathcal{C})$, its restriction to a pattern P is denoted by $\mathcal{C}_{|P}$.

The *translation* $u \cdot \mathcal{C}$ of a coloring $\mathcal{C}$ by a vector u is the coloring with support $\mathrm{Sup}(\mathcal{C}) + u$ such that, for all $z \in \mathrm{Sup}(\mathcal{C})$, it holds $(u \cdot \mathcal{C})(z + u) = \mathcal{C}(z)$. The *disjoint union* $\mathcal{C} \oplus \mathcal{C}'$ of two colorings $\mathcal{C}$ and $\mathcal{C}'$ is the coloring with support $\mathrm{Sup}(\mathcal{C}) \cup \mathrm{Sup}(\mathcal{C}')$ such that, for all $z \in \mathrm{Sup}(\mathcal{C})$, it holds $\mathcal{C} \oplus \mathcal{C}'(z) = \mathcal{C}(z)$ and for all $z \in \mathrm{Sup}(\mathcal{C}')$, it holds $\mathcal{C} \oplus \mathcal{C}'(z) = \mathcal{C}'(z)$. Colorings and their operations are depicted on Fig. 3.



(a) a coloring $\mathcal{C}$ (b) $\binom{1}{1} \cdot \mathcal{C}$ (c) $\mathcal{C} \oplus \binom{-2}{-2} \cdot \mathcal{C}$

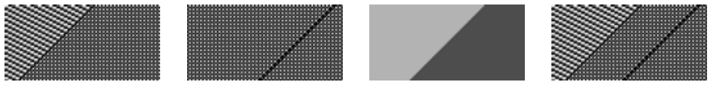
Fig. 3 Colorings, translations and disjoint unions

A *tiling constraint* is a pair (V, Υ) where V is a finite pattern and Υ is a subset of Σ^V. A coloring $\mathcal{C}$ *satisfies* a tiling constraint (V, Υ) if for each vector $u \in \mathbb{Z}^2$ such that V is a subset of $\mathrm{Sup}(u \cdot \mathcal{C})$, it holds $(u \cdot \mathcal{C})_{|V} \in \Upsilon$. For now on we fix a tiling constraint (V, Υ). A *tiling* is a coloring with support $\mathbb{Z}^2$ that satisfies the tiling constraint. For any pattern P, the *neighborhood* along the constraint (V, Υ) is defined as $\partial P = P \cup \{p + v | p \in P \text{ and } v \in V\}$.

In the following, for geometrical considerations, we will implicitly use variations of discrete forms of the Jordan curve theorem [15]. Two points $\binom{x}{y}, \binom{x'}{y'} \in \mathbb{Z}^2$ are *4-connected* if $\binom{|x-x'|}{|y-y'|} \in \left\{\binom{1}{0}, \binom{0}{1}\right\}$, *8-connected* if $\binom{|x-x'|}{|y-y'|} \in \left\{\binom{1}{0}, \binom{0}{1}, \binom{1}{1}\right\}$. A pattern P is *4-connected*, *resp. 8-connected*, if for each pair of points $z, z' \in P$, there exists a 4-connected, *resp.* 8-connected, path of points of P from z to z'. The discrete Jordan curve theorem states that any non empty 4-connected closed path separates the plane into two 8-connected patterns, the interior and exterior of the path. More generally, a *frontier* is a 4-connected pattern separating the plane into n 8-connected patterns, its *borders*.

3 Catenation of tilings

Let (V, Υ) be a tiling constraint and C a set of colorings satisfying this constraint. To generate tilings by catenating colorings in C, the idea is to construct a patchwork of colorings by cutting portions of coloring and glue them together so that tiling constraints are preserved. A simple patchwork of 2 tilings is depicted on Fig. 4.



(a) coloring ■ (b) coloring ■ (c) blueprint (d) patchwork

Fig. 4 A patchwork

Definition 1. A *patchwork* is a tiling $\mathcal{T}_\phi$ defined for each $z \in \mathbb{Z}^2$ by $\mathcal{T}_\phi(z) = \phi(z)(z)$ where $\phi : \mathbb{Z}^2 \to C$ is the *blueprint* of the patchwork such that:

1. $\forall \mathcal{C} \in C, \quad \partial\phi^{-1}(\mathcal{C}) \subseteq \mathrm{Sup}(\mathcal{C})$;
2. $\forall z \in \mathbb{Z}^2, \forall v \in V, \quad \phi(z)(z+v) = \phi(z+v)(z+v)$.

Patchworks provide a convenient way to combinatorially generate tilings from a set of valid colorings without knowing explicitly the tiling constraint: it is sufficient to know a super-set of the tiling neighborhood V and to cut colorings on a big enough boundary containing the same letters.

Topology is a classical tool of symbolic dynamics [11], tilings being exactly the shifts of finite type for two-dimensional words. The set of colorings is endowed with the so called Cantor topology: the product of the discrete topology on $\Sigma \cup \{\perp\}$ where $\perp$ denotes undefined color. This topology is compatible with the following distance on colorings: $d(\mathcal{C}, \mathcal{C}') = 2^{-\min\left\{|z|, \mathcal{C}(z) \neq \mathcal{C}'(z)\right\}}$. Let $\mathcal{O}_\mathcal{C}$ be the set of colorings $\mathcal{C}'$ such that $\mathcal{C}'_{|\mathrm{Sup}(\mathcal{C})} = \mathcal{C}_{|\mathrm{Sup}(\mathcal{C})}$. The set of $\mathcal{O}_\mathcal{C}$ for colorings $\mathcal{C}$ with a finite support is a base of clopen sets for the given compact perfect topology.

Proposition 1. *The set of patchworks over C is a compact set. Furthermore, it contains the tilings of the closure of C.*

Proof. Let $\mathcal{T}_i$ be a sequence of patchworks over C converging to a limit tiling $\mathcal{T}$. Consider the blueprints ϕ_i of these patchworks. For each cell $z \in \mathbb{Z}^2$, let v_z be the element $(-z \cdot \mathcal{T})_{|V}$ of Υ. Let $\phi(z)$ be any $\phi_i(z)$ such that $(-z \cdot \phi_i(z))_{|V} = v_z$ – such a $\phi_i(z)$ always exists by definition of patchworks as $\mathcal{T}_i$ converges to $\mathcal{T}$. The map ϕ is a blueprint for $\mathcal{T}$.

Let $\mathcal{C}_i$ be a sequence of colorings in C converging to a limit tiling $\mathcal{T}$. For each $\mathcal{C}_i$, let P_i be the largest pattern, for inclusion, such that $\mathcal{C}_{i|P_i} = \mathcal{T}_{|P_i}$. As the sequence $\mathcal{C}_i$ converges to $\mathcal{T}$, the sequence P_i converges to $\mathbb{Z}^2$. Without loss of generality, consider that P_i is an increasing sequence of patterns. For each i let $\delta(i)$ be the smallest j such that $\partial P_i \subseteq P_j$. Consider $P'_n = P_{\delta^n(1)}$, an increasing sub-sequence of P_i. Construct a blueprint ϕ as follows: for all $z \in \mathbb{Z}^2$, let $\phi(z) = P'_{\min\{n | z \in P'_n\}}$. By construction, this blueprint is valid and its patchwork is $\mathcal{T}$. ■

Corollary 1. *Let $\mathcal{O}_i$ be a base of open sets of colorings and C be a set of colorings containing at least one element of each $\mathcal{O}_i$. The set of patchworks over C is the whole set of tilings.* ■

In particular, the set of tiling constraints Υ, viewed as colorings, generates the whole set of tilings. The larger set of colorings with finite support generates the whole set of tilings. But this approach is heterogeneous: we combine colorings to obtain tilings. Can we restrict ourselves to combinations of tilings? More precisely, given a tiling constraint, can we recursively construct a recursive family of tilings T such that the set of patchworks over T is the whole family of tilings?

In the case of one-dimensional tilings, replacing $\mathbb{Z}^2$ by $\mathbb{Z}$, it is straightforward that the set of ultimately periodic tilings generates the whole set of tilings: the set of ultimately periodic tilings is a dense set – from any tiling $\mathcal{T}$ and any finite pattern P, one can construct an ultimately periodic tiling $\mathcal{T}'$ such that $\mathcal{T}_{|P} = \mathcal{T}'_{|P}$. In the case of two-dimensional tilings, due to the undecidability of the tiling problem [1, 19], there exists no such family. This result prohibits us to obtain a recursive set of tilings whose closure under catenation give us the whole set of tilings. Therefore, in the rest of the paper, we search for simplicity rather than being exhaustive.

4 Ultimately periodic tilings

Bi-periodic tilings are among the most regular ones and correspond to the idea of a *background* for cellular automata: a tiling $\mathfrak{B}$ with two non-co-linear periodicity vectors u and v such that $\mathfrak{B} = u \cdot \mathfrak{B} = v \cdot \mathfrak{B}$. As backgrounds are objects of dimension 2, if one wants to mix several backgrounds in a same tiling, the interface between two background is of dimension 1. The most regular kind of interface corresponds to the idea of a *particle*: a tiling $\mathfrak{P}$ with two non-co-linear vectors, the period u of the particle such that $\mathfrak{P} = u \cdot \mathfrak{P}$ and the period v of its backgrounds such that for all position $z \in \mathbb{Z}^2$, the extracted one-dimensional word $(\mathfrak{P}(z + vi))_{i \in \mathbb{Z}}$ is ultimately periodic. Of course, several particles might meet on the plane, leading to objects of dimension 0 that correspond to the idea of a *collision*. In this paper, an ultimately periodic tiling of the plane is such a collision.

Let $\sphericalangle_v(u, u')$ denote the angular portion of the plane, on the right hand side of u, starting in position $v \in \mathbb{Z}^2$ and delimited by the vectors $u, u' \in \mathbb{Z}^2$.

Formally, one might geometrically define a collision as follows (and depicted on Fig. 5):

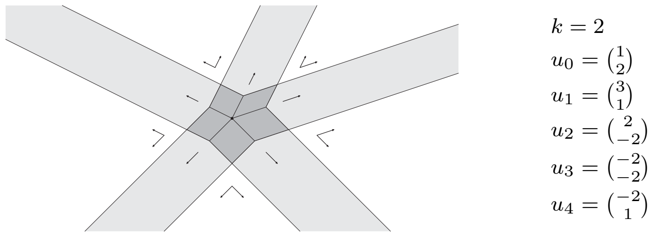


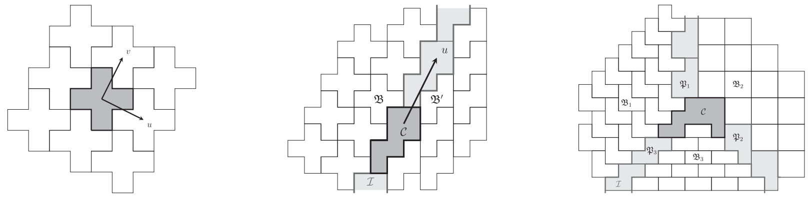
Fig. 5 Defining collisions through vectors

Definition 2. A *collision* is a tiling $\mathfrak{C}$ for which there exists an integer k and a finite cyclic sequence of n vectors $(u_i) \in \left(\mathbb{Z}^2\right)^{\mathbb{Z}_n}$ such that, for all $i \in \mathbb{Z}_n$, $\mathfrak{C}$ is u_i-periodic in z, *i.e.* $\mathfrak{C}(z) = \mathfrak{C}(z+u_i)$, for all positions z inside $\lhd_{ku_i}(u_{i-1}, u_{i+1})$.

Although it corresponds to intuition, this definition made it difficult to effectively use collisions in constructions since it does not identify components of the collision. To overcome this problem, we introduce *constructive* versions of collisions. Ideas behind such definitions is that all elements can be represented with a finite description. A background is entirely determined by two non-collinear vectors of periodicity u and v and by a coloring of finite support $\mathcal{C}$ that tiles the plane along u and v (i.e.$\bigoplus_{i,j\in\mathbb{Z}^2}(iu+jv)\cdot\mathcal{C}$ is a tiling) (see Fig. 6). Such a triple $(\mathcal{C}, u, v)$ is called *background representation.*

The same way, in a particle, the uni-periodic part can be characterised by a vector u and a coloring with finite support $\mathcal{C}$ which repeats along u ($\mathcal{I} = \bigoplus_{k\in\mathbb{Z}} ku\cdot\mathcal{C}$) is a frontier with two borders (L and R). The rest of particle can be described using two backgrounds $\mathfrak{B}$ and $\mathfrak{B}'$. The resulting coloring $\mathfrak{P} = \mathfrak{B}_{|L} \oplus \mathcal{I} \oplus \mathfrak{B}'_{|R}$ is require to be a tiling. Furthermore, we require to have a condition ensuring that the different portion have some common “safety zone”. This is done by adding the constraint that the function: $\phi : z \to \begin{cases} \mathfrak{P} & \text{if } z \in \mathrm{Sup}(I) \\ \mathfrak{B} & \text{if } z \in L \\ \mathfrak{B}' & \text{if } z \in R \end{cases}$ is the blueprint of a patchwork. Such a tuple $(\mathfrak{B}, \mathcal{C}, u, \mathfrak{B}')$ is called *particle representation.*

For collisions, the idea is basically the same (see Fig. 6) , the characterisation is based on a coloring with finite support $\mathcal{C}$ for the non-periodic part and a finite list of particles. Each particle defines a half-line starting form the center of the collision. The support of all the particles and the center must form a star and each consecutive pair of particles must have a common background to fill the space between them. Some safety zone is also required as in particle. This is formalised in the following definition:



(a) a background (b) a particle (c) a collision

Fig. 6 Principle of construction

Definition 3. A *collision representation* is a pair $(\mathcal{C}, L)$ where $\mathcal{C}$ is a finite pattern, L is a finite sequence of n particles $\mathfrak{P}_i = (\mathfrak{B}_i, \mathcal{C}_i, u_i, \mathfrak{B}'_i)$, satisfying:

1. $\forall i \in \mathbb{Z}_n, \quad \mathfrak{B}'_i = \mathfrak{B}_{i+1}$;
2. the support of $\mathcal{I} = \mathcal{C} \oplus \bigoplus_{i \in \mathbb{Z}_n, k \in \mathbb{N}} ku_i \cdot \mathcal{C}_i$ is a frontier with n borders;
3. For all $i \in \mathbb{Z}_n$, the support of $\mathcal{C} \oplus \bigoplus_{k \in \mathbb{N}} (ku_i \cdot \mathcal{C}_i \oplus ku_{i+1} \cdot \mathcal{C}_{i+1})$ is a frontier with two borders: let P_i be the border on the right of $\mathfrak{P}_i$;
4. $\mathfrak{C} = \mathcal{I} \oplus \bigoplus_i \mathfrak{B}_{i|P_i}$ is a tiling;
5. the function $\phi : z \to \begin{cases} \mathfrak{C} & \text{if } z \in \mathrm{Sup}(\mathcal{C}) \\ \mathfrak{P}_i & \text{if } z \in \mathrm{Sup}(\bigoplus_{k \in \mathbb{N}} ku_i \cdot \mathcal{C}_i) \\ \mathfrak{B}_i & \text{if } z \in P_i \end{cases}$ is the blueprint of a patchwork.

The set $\mathrm{Sup}(\mathcal{C})$ is called *perturbation* of the collision and $\mathrm{Sup}(\bigoplus_{k \in \mathbb{N}} ku_i \cdot \mathcal{C}_i)$ are called *perturbation* of the particle P_i.

The constructive definitions of particles, backgrounds and collisions provide us with a finite representation that allows us to recursively manipulate them. Contrary to intuition, representations are not invariant by translation. This seems unavoidable since we want to have means of expressing the relative position between two such representations. In the rest of the paper, we will always assume that background, particles and collisions are given by a representation.

5 Finite catenations

A blueprint of finitely many collisions might produce a tiling which is not a collision, however if the blueprint of the patchwork consists of finitely many 8-connected components, the patchwork is a collision. Using representations of collisions, a more regular family of patchworks can be defined: a catenation induces a patchwork combining collisions by binding pairs of similar particles as depicted on Fig. 2.

To "bind" collisions using particles, we need two identical particles facing each other such that the gap between them correspond to a integer number of particles n. Two particles $\mathfrak{P} = (\mathfrak{B}, \mathcal{C}, u, \mathfrak{B}')$ and $\tilde{\mathfrak{P}} = (\tilde{\mathfrak{B}}, \tilde{\mathcal{C}}, \tilde{u}, \tilde{\mathfrak{B}}')$ form a

n*-binding* if $\tilde{u} = -u$ (particles are facing each other), $\tilde{\mathcal{C}} = (n-1)u \cdot \mathcal{C}$ (they have the same finite pattern and gap is n repetitions) , $\tilde{\mathfrak{B}} = (n-1)u \cdot \mathfrak{B}'$, $\tilde{\mathfrak{B}}' = (n-1)u \cdot \mathfrak{B}$ (backgrounds are the same). The set $\bigoplus_{0<i<n-1} iu \cdot \mathcal{C}$ is called the *perturbation* of the n-binding.

Since we want to get rid of positions, we introduce the *potential* n*-binding.* The idea is that given two collisions and one particle for each collision, the particles $\mathfrak{P}_1$ and $\mathfrak{P}_2$ form a *potential* n*-binding* if up to a translation z, the two particles form an n-binding (i.e. $\mathfrak{P}_1$ and $z \cdot \mathfrak{P}_2$ form a n-binding). One can remark in case of potential n-binding, the translation vector z is unique.

Now the idea is that we can use potential n-binding to construct patchworks since background is bi-periodic and does not cause heavy harm for checking properties on it. The description needs to have collisions as points and particles as lines. Particles can be half-infinite (if they are not part of potential n-binding) or link two collisions. Since we work in the plane, it is sound to require that the constructed element is planar and that the order of particles is compatible with the collisions. At last, we add a connected condition to avoid problem with free parts of the map. This leads to the following definition:

Definition 4. A *catenation* is a connected planar map where:

- vertices are labeled by collisions;
- edges are potentially semi infinite;
- edges extremities are labeled by particles;
- edges order in a vertex is compatible with the order on particles in the corresponding collision.
- finite edges (of extremities $\mathfrak{P}_1$ and $\mathfrak{P}_2$) are labeled with an integer n such that $\mathfrak{P}_1$ and $\mathfrak{P}_2$ form a potential n-binding.

At this point, we want to transform the catenation into a patchwork. For this, let us first study some necessary conditions. Since we deal with a planar map, it is possible to define faces as elements of the dual of catenation. To transform a catenation into a patchwork, it is necessary that every face can be transformed into a patchwork. Since we have potential n_i-bindings, the translation induced between two consecutive collisions is fixed. Since the sequence of collisions in a face is cyclic, it is sound to require that the sequence of corresponding translation sum up to zero when cycling. This will be the first condition. Now, with this condition, it is possible to assign (up to a global constant) a translation to every collision such that all edges are n_i-bindings. With those objects, the basic idea is to construct a patchwork that corresponds to each collision, particle or n_i-binding on its perturbation. This implies that all perturbations does not enforce contradictions. One easy way to get rid of this risk is just to require that all perturbations are distinct (this will be our second condition). If these conditions are met then we speak of *valid* catenation.

Proposition 2. *It is possible to associate a patchwork (and therefore a space-time diagram) to every valid catenation.*

Proof. To prove this result, we shall give a potential blueprint and show that it satisfies the conditions. First of all, the condition on null translation after a round on every faces induce a unique set of translation (up to a constant) for every collision in the map since the map is connected. At this point, let us consider the collisions with those translations.

The second condition ensure that perturbations of collisions, bindings and particles are disjoint. Thus it is possible to define a blueprint linking any point of such a perturbation to the corresponding collision, particle or binding. Let us now study the points that are not mapped. Since the map is planar and particle (and also bindings) are isolating, every left point belongs to one unique face. On this face, the associated background with particles or collision or bindings present is unique (bindings ensure that two consecutive collisions are the same and collisions ensure this for consecutive particles and bindings). So we map those points to the corresponding background.

The last point is to show that the constructed blueprint does really satisfy the properties for patchwork. The first condition on definition is trivial since the used valid coloration are tilings. Let us go now to the second and main point.

For this last part, let us study the different cases. For example, if we are in a collision $\mathfrak{C}$ perturbation. If the neighborhood is also in $\mathfrak{C}$ perturbation or in perturbation of binding, particle belonging to $\mathfrak{C}$ or even of background with this property, then the neighborhood is by definition equal to the original one of a collision. the only difficult case is when in the neighborhood, there is a perturbation originated from another element. For example let us suppose this elements is in the perturbation from $\mathfrak{C}'$. In this case, in $\mathfrak{C}$ we have in these points some backgrounds or particles. But since perturbations do not overlap, we are in the border of $\mathfrak{C}'$. As we have requested in our constructive version representation to be patchworks, the border of $\mathfrak{C}'$ does correspond to the value of backgrounds or particles present in $\mathfrak{C}'$. By definition of our catenations, the backgrounds and particles are the same so elements of $\mathfrak{C}'$ are the same of those in $\mathfrak{C}$.
The same arguments do also apply for other cases thus ending the proof. ■

At this point, we have both a set of "simple" tilings (the collisions) and an operation generating new tilings from this set (the valid catenation). Despite being intuitive, catenations require to give explicitly the relative positions of collision via the number of repetitions of particles. Intuitively, we would like to give only the collisions involved and their organisation (as in Fig. 2c). With this approach, it is possible to define an alternative to catenation that does not require the number of repetitions to be given. The resulting element is called catenation scheme. Formally, a *catenation scheme* is a catenation whose label on finite edge where erased. Conversely, to go back from a catenation scheme to a catenation, one need to give every finite edge a label. Such elements of $\mathbb{N}^F$ where F is the set finite edges of the catenation scheme is called *affectation*. Moreover, it is called *valid* affectation if the resulting catenation is valid.

For a given catenation scheme, one natural question is whether it correspond to a tiling. To bring an answer one idea is to search for valid affectation of the scheme. In case of finite catenation scheme, we can achieve a very strong characterisation of this set and even compute it.

Theorem 1. *The set of valid affectation of a finite catenation is a recursive semi-linear set (i.e. a finite union of linear sets).*

Proof. To prove the main theorem, we will show that being a valid affectation of finite catenation scheme can be expressed with a formula in Presburger arithmetic (i.e first order logic on integer with addition and comparison). Since the set of solutions of formula in such arithmetic is a recursive semi-linear set [10] this will conclude the proof. One can note that the construction of the solution is explicit even if the complexity is non-elementary.

In our formula, the number of repetitions of each finite edge will correspond to free variables. let us call them $r_1, \ldots, r_n$. Since the conditions for valid catenation are for each face, the global formula F will consists on the conjunction of an elementary formula for each face: $F = \wedge_{f\text{face}} F_f$. For each face, let us look at the two conditions. First one (going back to the same point after a turn around the face) can be easily expressed: the translation induced by a particle i is just r_i times the vector of repetition of the particle u_i (just note that the direction of the particles is chosen in the face) which is a known constant. For the translation induced by collision ϵ_c they are know constant. So the formula is on the form $F_{f,1} = \Sigma_{i\text{particles in the face}} u_i r_i + \Sigma_{c\text{collisions}} \epsilon_c = \binom{0}{0}$. For the second condition (non overlap of perturbation) it can be expressed with the conjunction that any pair of points of different perturbations are distinct. In the case of collision perturbation, it is trivial since there is only a finite (and known) number of perturbation points. For bindings, it is more difficult since the set of points can be expressed with a universal quantifier with the following remark, the set of points in the binding's perturbation correspond to the set of points of the particle perturbation $\mathrm{Sup}(\mathcal{C}_i)$ (a finite number) for every integer n multiple of the vector of repetition u_i which is between 0 and the number of repetition r_i. thus the formula is on the form: $\forall x, 0 < x < r_i \Rightarrow \wedge_{p \in \mathrm{Sup}(\mathcal{C}_i)} p + u_i r_i \neq z$ where z are points for the other considered perturbation. The same applies for free particles (just omit the upper bound in the comparison).

With this, we have show how to construct the Presburger formula which conclude the proof. ■

With this theorem we achieve a very strong framework for cellular automata. After have extracted a set of collisions, one can give the desired finite catenation scheme and automatically check the necessary and sufficient conditions for that scheme to exists. This method would make proves far more understandable and could avoid the need to rely on combinatorial proves to ensure validity of intuition. For now, the main limitation of those results are that only the field of finite catenations are treated. One main goal of future work is to achieve such kind of result for infinite catenation schemes. Due to the infinite nature of such elements, such strong a characterisation is excluded but we hope to have sufficient computable conditions for affectation of a wide range of “regular” infinite catenations.

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