Topological Derivatives in Plane Elasticity

Jan Sokołowski and Antoni Żochowski

Abstract We present a method for construction of the topological derivatives in plane elasticity. It is assumed that a hole is created in the subdomain of the elastic body which is filled out with isotropic material. The asymptotic analysis of elliptic boundary value problems in singularly perturbed geometrical domains is used in order to derive the asymptotics of the shape functionals depending on the solutions to the boundary value problems. Our method allows for the asymptotic expansions of arbitrary order, since the explicit solutions to the boundary value problems are obtained by the method of elastic potentials. Some numerical results are presented to show the applicability of the proposed method in numerical analysis of elliptic problems.

1 Introduction

One of the most important applications of the topological derivatives of shape functionals is elasticity, in particular in the fields of optimal design in structural mechanics and the numerical solution for inverse problems of detection of small imperfections. The mathematical theory of asymptotic analysis of elliptic boundary value problems in singularly perturbed domains, is considered in [6] and [10]. The method of compound asymptotic expansions in the framework of the asymptotic analysis leads to the asymptotic expansions of solutions and to the topological derivatives of the shape functionals as it is described in details, e.g., in the paper [12] for bound-

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ary value problems of linearized elasticity. The concept of topological derivatives of shape functionals [18] is derived in the framework of the method of compound asymptotic expansions [10], one of the techniques used in the asymptotic analysis of the boundary value problems in singularly perturbed geometrical domains. The so-called truncation method is described, e.g., in [9] (see [3] for further developments). The asymptotic analysis in impedance imaging and the theory of composite materials can be found, e.g., in [1].

We present here the results on asymptotics of the shape functionals for the specific class of the elliptic boundary value problems. Let there be given an elastic body which occupies the reference domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$, with the material properties defined by the Hooke's tensor $\mathscr{C}_{ijkl}$, $i,j,k,l = 1,\dots,d$. We assume that there is a ball $B_R(x) \subset \Omega$, $R > 0$, with the center $x \in \Omega$, filled with an isotropic material characterized by its Lame coefficients λ, μ. We investigate the asymptotics for $\rho \to 0$ of the displacement and the stress fields in the body $\Omega_\rho = \Omega \setminus \overline{B_\rho(x)}$ due to the creation of a small hole $B_\rho(x) \subset B_R(x)$ of the radius $R > \rho \to 0$. We also perform the asymptotic analysis of some shape functionals depending on the solution of the elasticity boundary value problems in $\Omega_\rho = \Omega \setminus \overline{B_\rho(x)}$ for $\rho \to 0$. It seems that the imposed condition on the isotropy of $B_R(x)$ cannot be avoided since for the specific application of the existing methods of asymptotic analysis we need the knowledge of fundamental solution of the elliptic operator in the region $B_R(x)$. In order to obtain the required asymptotics in the whole domain we employ [21] a domain decomposition technique combined with the fine analysis of the properties of the Steklov-Poincaré operator $\mathscr{A}_\rho$, $\rho \geq 0$, defined in the ball $B_R(x)$ as well as in the ring $C(R,\rho) = B_R(x) \setminus \overline{B_\rho(x)}$.

The paper contains the complete mathematical tools which are used to derive the form of topological derivatives for the specific class of composite elastic materials in two spatial dimensions.

2 Topological Derivatives of Shape Functionals in Isotropic Elasticity

We are going to present the results which can be obtained for 2D boundary value problems of linear elasticity. The results for 3D are not in the same explicit form. The same type of results on topological derivatives is derived for the contact problems by means of the asymptotic analysis combined with the domain decomposition technique [21].

We briefly introduce the concept of the topological derivative for an arbitrary shape functional. The topological derivative denoted by $\mathscr{T}_\Omega$ of a shape functional $\mathscr{J}(\Omega)$ is introduced in [18] in order to characterize the infinitesimal variation of $\mathscr{J}(\Omega)$ with respect to the infinitesimal variation of the topology of the domain Ω. The topological derivative allows us to derive the new optimality condition in the interior of an optimal domain, if such a domain exists and if the shape functional under studies admits the topological derivatives, for the shape optimization problem:

$$\mathscr{J}(\Omega^*) = \inf_{\Omega} \mathscr{J}(\Omega). \tag{1}$$

The optimal domain Ω^* is characterized by the first order condition [17] defined on the boundary of the optimal domain Ω^*, $dJ(\Omega^*;V) \geq 0$ for all admissible vector fields V, and by the following optimality condition defined in the interior of the domain Ω^*:

$$\mathscr{T}_{\Omega^*}(x) \geq 0 \text{ in } \Omega^*. \tag{2}$$

The other use of the topological derivative is connected with approximating the influence of the holes in the domain on the values of integral functionals of solutions, which allows us, e.g., to solve a class of shape inverse problems.

In general terms the notion of the *topological* derivative (TD) has the following meaning. Assume that $\Omega \subset \mathrm{I\!R}^N$ is an open set and that there is given a shape functional

$$\mathscr{J} : \Omega \setminus K \to \mathrm{I\!R} \tag{3}$$

for any compact subset $K \subset \overline{\Omega}$. We denote by $B_\rho(x), x \in \Omega$, the ball of radius $\rho > 0$, $B_\rho(x) = \{y \in \mathrm{I\!R}^N \,|\, \|y - x\| < \rho\}$, $\overline{B_\rho(x)}$ is the closure of $B_\rho(x)$, and assume that there exists the following limit

$$\mathfrak{T}(x) = \lim_{\rho \downarrow 0} \frac{\mathscr{J}(\Omega \setminus \overline{B_\rho(x)}) - \mathscr{J}(\Omega)}{|\overline{B_\rho(x)}|}. \tag{4}$$

The function $\mathfrak{T}(x), x \in \Omega$, is called the topological derivative of $\mathscr{J}(\Omega)$, and provides the information on the infinitesimal variation of the shape functional $\mathscr{J}$ if a small hole is created at $x \in \Omega$. This definition is suitable for Neumann–type boundary conditions on ∂B_ρ.

In many cases this characterization is constructive [5, 2, 3, 8, 12, 14, 15], i.e. TD can be evaluated for shape functionals depending on solutions of partial differential equations defined in the domain Ω.

2.1 Problem Setting for Elasticity Systems

We introduce the elasticity system in a form convenient for the evaluation of topological derivatives. Let us consider the elasticity equations in $\mathrm{I\!R}^N$, where $N = 2$ for 2D and $N = 3$ for 3D,

$$\begin{cases} \operatorname{div} \sigma(u) = 0 \;\text{ in }\; \Omega \\ u = g \;\text{ on }\; \Gamma_D \\ \sigma(u)n = T \;\text{ on }\; \Gamma_N \end{cases} \tag{5}$$

and the same system in the domain with the spherical cavity $B_\rho(x_0) \subset \Omega$ centered at $x_0 \in \Omega$, $\Omega_\rho = \Omega \setminus \overline{B_\rho(x_0)}$,

$$\begin{cases} \operatorname{div} \sigma_\rho(u_\rho) = 0 \text{ in } \Omega_\rho \\ u_\rho = g \text{ on } \Gamma_D \\ \sigma_\rho(u_\rho)n = T \text{ on } \Gamma_N \\ \sigma_\rho(u_\rho)n = 0 \text{ on } \partial B_\rho(x_0) \end{cases} \tag{6}$$

where n is the unit outward normal vector on $\partial\Omega_\rho = \partial\Omega \cup \partial B_\rho(x_0)$. Assuming that $0 \in \Omega$, we can consider the case $x_0 = 0$.

Here u and u_ρ denote the displacement vectors fields, g is a given displacement on the fixed part Γ_D of the boundary, t is a traction prescribed on the loaded part Γ_N of the boundary. In addition, σ is the Cauchy stress tensor given, for $\xi = u$ (eq. 5) or $\xi = u_\rho$ (eq. 6), by

$$\sigma(\xi) = D\nabla^s\xi, \tag{7}$$

where $\nabla^s(\xi)$ is the symmetric part of the gradient of vector field ξ, that is

$$\nabla^s(\xi) = \frac{1}{2}\left(\nabla\xi + \nabla\xi^T\right), \tag{8}$$

and D is the elasticity tensor,

$$D = 2\mu I\!I + \lambda\,(I \otimes I), \tag{9}$$

with

$$\mu = \frac{E}{2(1+\nu)}, \;\; \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \;\text{ and }\; \lambda = \lambda^* = \frac{\nu E}{1-\nu^2}, \tag{10}$$

E being the Young's modulus, ν the Poisson's ratio and λ^* the particular case for plane stress. In addition, I and $I\!I$ respectively are the second and fourth order identity tensors. Thus, the inverse of D is

$$D^{-1} = \frac{1}{2\mu}\left[I\!I - \frac{\lambda}{2\mu + N\lambda}\,(I \otimes I)\right]. \tag{11}$$

The first shape functional under consideration depends on the displacement field,

$$J_u(\rho) = \int_{\Omega_\rho} F(u_\rho)\,d\Omega, \;\; F(u_\rho) = (Hu_\rho \cdot u_\rho)^p, \tag{12}$$

where F is a C^2 function. It is also useful for further applications in the framework of elasticity to introduce the yield functional of the form

$$J_\sigma(\rho) = \int_{\Omega_\rho} S\sigma(u_\rho)\cdot\sigma(u_\rho)\,d\Omega, \tag{13}$$

where S is an isotropic fourth-order tensor. Isotropicity means here that S may be expressed as follows

$$S = 2mI\!I + l\,(I \otimes I), \tag{14}$$

where l, m are real constants. Their values may vary for particular yield criteria. The following assumption assures that J_u, J_σ are well defined for solutions of the elasticity system.

(**CONDITION A**) The domain Ω has piecewise smooth boundary, which may have reentrant corners with $\alpha < 2\pi$ created by the intersection of two planes. In addition, g, t must be compatible with $u \in H^1(\Omega; \mathbb{R}^N)$.

The interior regularity of u in Ω is determined by the regularity of the right hand side of the elasticity system. For simplicity the following notation is used for functional spaces,

$$H^1_g(\Omega_\rho) = \{\psi \in [H^1(\Omega_\rho)]^N \mid \psi = g \text{ on } \Gamma_D\}, \tag{15}$$

$$H^1_{\Gamma_D}(\Omega_\rho) = \{\psi \in [H^1(\Omega_\rho)]^N \mid \psi = 0 \text{ on } \Gamma_D\}, \tag{16}$$

$$H^1_{\Gamma_D}(\Omega) = \{\psi \in [H^1(\Omega)]^N \mid \psi = 0 \text{ on } \Gamma_D\}, \tag{17}$$

here we use the convention that, e.g., $H^1_g(\Omega_\rho)$ stands for the Sobolev space of vector functions $[H^1_g(\Omega_\rho)]^N$.

The weak solutions to the elasticity systems are defined in the standard way.

Find $u_\rho \in H^1_g(\Omega_\rho)$ such that, for every $\phi \in H^1_{\Gamma_D}(\Omega)$,

$$\int_{\Omega_\rho} D\nabla^s u_\rho \cdot \nabla^s \phi \, d\Omega = \int_{\Gamma_N} T \cdot \phi \, dS. \tag{18}$$

We introduce the adjoint state equations in order to simplify the form of shape derivatives of functionals J_u, J_σ. For the functional J_u they take on the variational form: Find $w_\rho \in H^1_{\Gamma_D}(\Omega_\rho)$,

$$\int_{\Omega_\rho} D\nabla^s w_\rho \cdot \nabla^s \phi \, d\Omega = -\int_{\Omega_\rho} F'_u(u_\rho) \cdot \phi \, d\Omega, \tag{19}$$

for every $\phi \in H^1_{\Gamma_D}(\Omega)$, whose Euler-Lagrange equation reads

$$\begin{cases} \text{div } \sigma_\rho(w_\rho) = F'_u(u_\rho) & \text{in } \Omega_\rho \\ w_\rho = 0 & \text{on } \Gamma_D \\ \sigma_\rho(w_\rho)n = 0 & \text{on } \Gamma_N \\ \sigma_\rho(w_\rho)n = 0 & \text{on } \partial B_\rho(x_0) \end{cases} \tag{20}$$

while $v_\rho \in H^1_{\Gamma_D}(\Omega_\rho)$ is the adjoint state for J_σ and satisfies for all test functions $\phi \in H^1_{\Gamma_D}(\Omega)$ the following integral identity:

$$\int_{\Omega_\rho} D\nabla^s v_\rho \cdot \nabla^s \phi \, d\Omega = -2\int_{\Omega_\rho} DS\sigma(u_\rho) \cdot \nabla^s \phi \, d\Omega, \tag{21}$$

whose associated Euler-Lagrange equation becomes

$$\begin{cases} \operatorname{div} \sigma_\rho(v_\rho) = -2\operatorname{div}\left(DS\sigma_\rho(u_\rho)\right) & \text{in } \Omega_\rho \\ v_\rho = 0 & \text{on } \Gamma_D \\ \sigma_\rho(v_\rho)n = -2DS\sigma_\rho(u_\rho)n & \text{on } \Gamma_N \\ \sigma_\rho(v_\rho)n = -2DS\sigma_\rho(u_\rho)n & \text{on } S_\rho(x_0) = \partial B_\rho(x_0) \end{cases} \tag{22}$$

Remark 0.1. We observe that DS can be written as

$$DS = 4\mu m I\!I + \gamma(I \otimes I) \tag{23}$$

where

$$\gamma = \lambda l N + 2(\lambda m + \mu l)\,. \tag{24}$$

Thus, when $\gamma = 0$, the boundary condition on $\partial B_\rho(x_0)$ in equation (22) becomes homogeneous and the yield criteria must satisfy the constraint

$$\frac{m}{l} = -\left(\frac{\mu}{\lambda} + \frac{N}{2}\right), \tag{25}$$

which is satisfied for the energy shape functional. In this particular case, tensor S is given by

$$S = \frac{1}{2}D^{-1} \;\; \Rightarrow \;\; \gamma = 0 \text{ and } 2m + l = \frac{1}{2E}, \tag{26}$$

which implies that the adjoint solution associated to J_σ can be explicitly obtained, such that $v_\rho = -(u_\rho - g)$.

2.2 Topological Derivatives in 2D Elasticity

We recall here the results derived in [18] for the 2D case. The principal stresses associated with the displacement field u are denoted by $\sigma_I(u)$, $\sigma_{II}(u)$, the trace of the stress tensor $\sigma(u)$ is denoted by $\operatorname{tr}\sigma(u) = \sigma_I(u) + \sigma_{II}(u)$. The shape functionals J_u, J_σ are defined in the same way as presented before, with the tensor S isotropic (that is similar to D). The weak solutions to the elasticity system as well as adjoint equations are defined in standard way. Then, from the expansions presented in the Appendix, we may formulate the following result [18]:

Theorem 1. *The expressions for the topological derivatives of the functionals J_u, J_σ have the form*

$$\begin{aligned} \mathscr{T} J_u(x_0) &= -\left[F(u) + \frac{1}{E}\left(a_u a_w + 2 b_u b_w \cos 2\delta\right)\right]_{x=x_0} \\ &= -\left[F(u) + \frac{1}{E}\left(4\sigma(u)\cdot\sigma(w) - \operatorname{tr}\sigma(u)\operatorname{tr}\sigma(w)\right)\right]_{x=x_0} \end{aligned} \tag{27}$$

$$\mathscr{T}J_\sigma(x_0) = -\left[\eta(a_u^2+2b_u^2)+\frac{1}{E}(a_u a_v+2b_u b_v\cos 2\delta)\right]_{x=x_0}$$
$$= -\Big[\eta(4\sigma(u)\cdot\sigma(u)-(\mathrm{tr}\sigma(u))^2) \tag{28}$$
$$+\frac{1}{E}(4\sigma(u)\cdot\sigma(v)-\mathrm{tr}\sigma(u)\mathrm{tr}\sigma(v))\Big]_{x=x_0}$$

Some of the terms in (27), (28) require explanation. According to equation (24) for $N=2$, constant η is given by

$$\eta = l+2\left(m+\gamma\frac{\nu}{E}\right). \tag{29}$$

Furthermore, we denote

$$\begin{aligned} a_u &= \sigma_I(u)+\sigma_{II}(u), & b_u &= \sigma_I(u)-\sigma_{II}(u),\\ a_w &= \sigma_I(w)+\sigma_{II}(w), & b_w &= \sigma_I(w)-\sigma_{II}(w),\\ a_v &= \sigma_I(v)+\sigma_{II}(v), & b_v &= \sigma_I(v)-\sigma_{II}(v). \end{aligned} \tag{30}$$

δ denotes the angle between principal stress directions for displacement fields u and w in (27), and for displacement fields u and v in (28).

Remark 0.2. For the energy stored in a 2D elastic body, tensor S is given by eq. (26), $\gamma = 0$ and $\eta = 1/(2E)$. Thus, since $v = -(u-g)$, we obtain the following well-known result

$$\mathscr{T}J_\sigma(x_0) = \frac{1}{2E}\left[4\sigma(u)\cdot\sigma(u)-(\mathrm{tr}\sigma(u))^2\right]_{x=x_0}. \tag{31}$$

3 Topological Derivatives for Contact Problems

In order to describe the domain decomposition method applied to the asymptotic analysis, and introduce the Steklov-Poincaré operators for the rings $C(R,\rho)$, $\rho \geq 0$, we present the related results for the two dimensional frictionless contact problems. Such problems are non smooth, therefore, in general, only the first term of the exterior asymptotic expansion of solutions can be derived. However, this leads to the topological derivatives of some shape functionals. We change the notation, compared to the previous sections, in particular $\mathbf{u}$ stands now for the displacement vector, and $\sigma(\mathbf{u})$ is the corresponding stress tensor.

We consider the isotropic two dimensional elasticity problem in plane stress formulation, the isotropy is in fact required only in the vicinity of a small hole. On a part Γ_u of $\partial\Omega$ we assume that the body is clamped $\mathbf{u}=0$, the part Γ_g is loaded $\sigma(\mathbf{u}).\mathbf{n}=\mathbf{g}$ and on the part Γ_c there is the frictionless contact

$$\begin{aligned} u_n \geq 0, &\qquad \sigma_n \leq 0,\\ \sigma_n u_n = 0, &\qquad \sigma_\tau = \sigma.\mathbf{n}-\sigma_n\mathbf{n} = 0. \end{aligned} \tag{32}$$

Here $u_n = u_i n_i$, $\sigma_n = n_i \sigma_{ij} n_j$, $\boldsymbol{\sigma}.\mathbf{n} = \{\sigma_{ij} n_j\}_{i=1,2}$. We define also the ring $C(R,\rho) = B(R) \setminus \overline{B(\rho)}$ with $R > \rho$ and such that $B(R) \subset \Omega$, as well as $\Omega(r) = \Omega \setminus \overline{B(r)}$.

For such a problem it is impossible to evaluate topological derivatives of shape functionals by means of adjoint variables without additional assumptions on the strict complementarity type for the unknown solution. Therefore, we propose a method for computing the perturbation, caused by the hole $B(\rho)$, of the solution itself.

The bilinear form corresponding to the elastic energy may be written as

$$a(\rho;\mathbf{u},\mathbf{v}) = \frac{1}{2}\int_{\Omega(\rho)} \sigma(\mathbf{u}) : \varepsilon(\mathbf{v})\, dx \tag{33}$$

($\sigma : \varepsilon = \sigma_{ij}\varepsilon_{ij}$) for $\mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega)$ and the work of external forces is

$$L(\mathbf{u}) = \int_{\Gamma_g} \mathbf{u}^\top \mathbf{g}\, ds. \tag{34}$$

The method of the domain decomposition type is based on the analysis of the Steklov-Poincaré operator $\mathscr{A}_\rho$ defined in the following way. Consider the boundary value problem

$$\mathscr{L}\mathbf{w} = 0 \ \text{ in } \ C(R,\rho), \ \ \sigma_n(\mathbf{w}) = 0 \ \text{ on } \ \partial B(\rho), \ \ \mathbf{w} = \mathbf{v} \ \text{ on } \ \partial B(R). \tag{35}$$

Then we set

$$\mathscr{A}_\rho \mathbf{v} = \sigma_n(\mathbf{w}) \ \text{ on } \ \partial B(R). \tag{36}$$

Thus $\mathscr{A}_\rho$ is a mapping

$$\mathscr{A}_\rho : \mathbf{H}^{1/2}(\partial B(R)) \ \mapsto \ \mathbf{H}^{-1/2}(\partial B(R)). \tag{37}$$

It can be demonstrated constructively that

$$\mathscr{A}_\rho = \mathscr{A}_0 + \rho^2 \mathscr{A}_1 + \rho^4 \mathscr{A}_2 + \ldots \tag{38}$$

in the linear operator norm corresponding to (37). Using this notation we have

$$a(\rho;u,u) = \frac{1}{2}\int_{\Omega(R)} \sigma(\mathbf{u}) : \varepsilon(\mathbf{u})\, dx + \frac{1}{2}\int_{C(R,\rho)} \sigma(\mathbf{u}) : \varepsilon(\mathbf{u})\, dx \tag{39}$$

as well as

$$\begin{aligned} \frac{1}{2}\int_{C(R,\rho)} \sigma(\mathbf{u}) : \varepsilon(\mathbf{u})\, dx &= \frac{1}{2}\langle \mathscr{A}_\rho \mathbf{u}, \mathbf{u}\rangle_{\partial B(R)} \\ &= \frac{1}{2}\langle \mathscr{A}_0 \mathbf{u}, \mathbf{u}\rangle_{\partial B(R)} + \frac{1}{2}\rho^2 \langle \mathscr{A}_1 \mathbf{u}, \mathbf{u}\rangle_{\partial B(R)} + \mathscr{R}(\mathbf{u},\mathbf{u}) \end{aligned} \tag{40}$$

where $\mathscr{R}(\mathbf{u},\mathbf{u})$ is of the order $O(\rho^4)$ on bounded sets in $\mathbf{H}^{1/2}(\partial B(R))$. With $\mathscr{A}_1$ we associate the bilinear form

$$b(\mathbf{u},\mathbf{v}) = \frac{1}{2}\langle \mathscr{A}_1\mathbf{u},\mathbf{u}\rangle_{\partial B(R)}. \tag{41}$$

It is sufficient to consider the following approximation of the energy bilinear form in order to construct one term exterior approximation of the solution to the contact problem

$$a(\rho;\mathbf{u},\mathbf{u}) := a(0;\mathbf{u},\mathbf{u}) + \rho^2 b(\mathbf{u},\mathbf{u}). \tag{42}$$

Denote by $\mathbf{H}^1_{\Gamma_u}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \mid \mathbf{v} = 0 \text{ on } \Gamma_u\}$ the Sobolev space, and let K be the convex cone

$$K = \{\mathbf{v} \in \mathbf{H}^1_{\Gamma_u}(\Omega) \mid v_n \geq 0 \text{ on } \Gamma_c\}. \tag{43}$$

Recall that the following variational inequality furnishes the weak solutions to our contact problem in $\Omega(\rho)$

$$\mathbf{u} \in K\colon\ a(\rho;\mathbf{u},\mathbf{u}-\mathbf{v}) \geq L(\mathbf{v}-\mathbf{u})\ \ \forall \mathbf{v} \in K. \tag{44}$$

Taking into account the approximation (42) and using abstract results on the differentiability of metric projection onto the polyhedric convex sets in Dirichlet space [16] we have the following result.

Theorem 2. *For ρ sufficiently small we have on $\Omega(R)$ the following expansion of the solution $\mathbf{u}$ with respect to the parameter ρ at $0+$,*

$$\mathbf{u} = \mathbf{u}_0 + \rho^2\mathbf{q} + o(\rho^2) \text{ in } \mathbf{H}^1(\Omega(R)), \tag{45}$$

where the topological derivative $\mathbf{q}$ of the solution $\mathbf{u}$ to the contact problem is given by the unique solution of the following variational inequality

$$\mathbf{q} \in \mathscr{S}_K(\mathbf{u})\colon\ a(0;\mathbf{q},\mathbf{v}-\mathbf{q}) + b(\mathbf{u},\mathbf{v}-\mathbf{q}) \geq 0\ \ \forall \mathbf{v} \in \mathscr{S}_K(\mathbf{u}), \tag{46}$$

where

$$\mathscr{S}_K(\mathbf{u}) = \{\mathbf{v} \in \mathbf{H}^1_{\Gamma_u}(\Omega) \mid v_n \leq 0 \text{ on } \Xi(\mathbf{u}),\ a(0;\mathbf{u},\mathbf{v}) = 0\}. \tag{47}$$

The coincidence set $\Xi(\mathbf{u}) = \{\mathbf{x} \in \Gamma_c \mid u_n(\mathbf{x}) = 0\}$ is well defined [16] for any $\mathbf{u} \in \mathbf{H}^1(\Omega)$, and $\mathbf{u}_0 \in K$ is the solution of (44) for $\rho = 0$.

4 Complex Variable Method

In order to find an exact form of the Steklov-Poincaré operator in plane elasticity we need an analytic form of the solution for the elasticity system in the ring, with general displacement condition on the outer boundary and traction free inner boundary, parameterized by the (small) inner radius ρ. Let us assume for simplicity that the center of the ring lies at origin of the coordinate system, and take polar coordinates (r,θ) with $\mathbf{e}_r$ pointing outwards and $\mathbf{e}_\theta$ perpendicularly in the counter-clockwise direction. Then the displacement on the outer boundary $r = R$ may be given in the form of a Fourier series

$$2\mu(u_r + iu_\theta) = \sum_{k=-\infty}^{k=+\infty} U_k e^{ik\theta}. \tag{48}$$

The regularity condition for the boundary data translate into some inequalities for coefficients U_k, as will be made precise later.

The solution in the ring must be compared with the solution in the full circle, so we will have to construct it as well. Probably the best tool for obtaining both exact solutions is the complex variable method, described in [11]. It states that for plane domains with one hole these solutions have the form

$$\begin{aligned} \sigma_{rr} - i\sigma_{r\theta} &= 2\Re\phi' - e^{2i\theta}(\bar{z}\phi'' + \psi'), \\ \sigma_{rr} + i\sigma_{\theta\theta} &= 4\Re\phi', \\ 2\mu(u_r + iu_\theta) &= e^{-i\theta}(\kappa\phi - z\bar{\phi}' - \bar{\psi}), \end{aligned} \tag{49}$$

where $\phi,\ \psi$ are given by complex series

$$\begin{aligned} \phi &= A\log(z) + \sum_{k=-\infty}^{k=+\infty} a_k z^k, \\ \psi &= -\kappa\bar{A}\log(z) + \sum_{k=-\infty}^{k=+\infty} b_k z^k. \end{aligned} \tag{50}$$

Here μ is the Lame constant, ν is the Poisson ratio, $\kappa = 3 - 4\nu$ in the plain strain case, and $\kappa = (3-\nu)/(1+\nu)$ for plane stress.

Now we can substitute displacement condition for $r = R$ into

$$\begin{aligned} 2\mu(u_r + iu_\theta) = {} & 2\kappa Ar\log(r)\frac{1}{z} - \bar{A}\frac{1}{r}z + \\ & + \sum_{p=-\infty}^{p=+\infty} [\kappa r a_{p+1} - (1-p)\bar{a}_{1-p} r^{-2p+1} - \bar{b}_{-(p+1)} r^{-2p-1}] z^p \end{aligned} \tag{51}$$

and obtain the infinite system of linear equations

$$\begin{aligned} p = -1: &\ 2\kappa Ar\log(r) + (\kappa a_0 - \bar{b}_0) - 2\bar{a}_2 r^2 = U_{-1} \\ p = 1: &\ -\bar{A} + \kappa r^2 a_2 - \bar{b}_{-2}\frac{1}{r^2} = U_1 \\ p \notin \{-1,1\}: &\ \kappa r^{p+1} a_{p+1} - (1-p)\bar{a}_{1-p} r^{-p+1} - \bar{b}_{-(p+1)} r^{-(p+1)} = U_p. \end{aligned} \tag{52}$$

The traction-free condition

$$\sigma.\mathbf{e}_r = [\sigma_{rr}, \sigma_{r\theta}]^\top \tag{53}$$

on some circle means $\sigma_{rr} = \sigma_{r\theta} = 0$. Hence, assuming $r := \rho$, we have another infinite system

$$
\begin{aligned}
p = -1:\ & 2A + 2\bar{a}_2 r^2 + 2\frac{1}{r^2} b_{-2} = 0 \\
p = 1:\ & (\kappa+1)\frac{1}{r^2}\bar{A} = 0 \\
p \notin \{-1,1\}:\ & (1+p)a_{p+1} + \bar{a}_{1-p} r^{-2p} + \frac{1}{r^2} b_{p-1} = 0.
\end{aligned} \tag{54}
$$

Denote $d_0 = \kappa a_0 - \bar{b}_0$ since a_0, b_0 appear only in this combination. Using (52) we may recover the solution for the full circle. Because in this case the singularities must vanish, we have $b_{-k} = a_{-k} = A = 0$ for $k = 1, 2, \dots$ and comparing the same powers of r:

$$
\begin{aligned}
&d_0^0 = U_{-1} + \frac{2}{\kappa}\bar{U}_1, \quad \Re a_1^0 = \frac{1}{(\kappa-1)R}\Re U_0, \quad \Im a_1^0 = \frac{1}{(\kappa+1)R}\Im U_0 \\
&a_k^0 = \frac{1}{\kappa R^k} U_{k-1}, \quad b_k^0 = -\frac{1}{R^k}[(k+2)\frac{1}{\kappa}U_{k+1} + \bar{U}_{-(k+1)}], \quad k > 1.
\end{aligned} \tag{55}
$$

Now let us repeat the same procedure for the ring. Now the singularities may be present, because 0 does not belong to the domain. Hence, from (52) for $r = R$ and (54) for $r = \rho$ we obtain $A = 0$ and the formulas

$$
\begin{aligned}
d_0 &= A_{-1} + \frac{2R^4}{\kappa R^4 + \rho^4}\bar{U}_1, & a_2 &= \frac{R^2}{\kappa R^4 + \rho^4} U_1 \\
\Re a_1 &= \frac{R}{(\kappa-1)R^2 + 2\rho^2}\Re U_0, & \Im a_1 &= \frac{1}{\kappa+1}\Im A_0 \\
b_{-1} &= -\frac{2\rho^2 R}{(\kappa-1)R^2 + 2\rho^2}\Re U_0, & b_{-2} &= -\frac{\rho^4 R^2}{\kappa R^4 + \rho^4}\bar{U}_1
\end{aligned} \tag{56}
$$

The rest of the coefficients will be computed later. However, we may at this stage compare the results with known solutions for the uniformly stretched circle or ring obtained in another way. In such a case $U_0 = 2\mu u_r(R)$ does not vanish and, for the full circle, $\psi = 0$, $\phi = a_1^0 z$ with

$$
a_1^0 = \frac{2\mu}{(\kappa-1)R} u_r(R). \tag{57}
$$

For the ring we have $\phi = a_1 z,\ \psi = b_{-1}\frac{1}{z}$ where

$$
a_1 = \frac{1}{(\kappa-1) + 2\rho^2} 2\mu u_R(1), \quad b_{-1} = -\frac{2\rho^2}{(\kappa-1) + 2\rho^2} 2\mu u_R(1). \tag{58}
$$

After substitutions we obtain, in both cases, the same results as given in [7]. Similarly the comparison with the solution for the ring with displacement conditions on both boundaries, obtained in [4] also using complex method, confirms the correctness of the formulas.

There remains to compute the rest of the coefficients a_k, b_k for the case of the ring. Taking $p = -k,\ k = 2, 3, \dots$ in conditions on both boundaries gives the system

$$\begin{aligned} \kappa a_{-(k-1)} R^{-(k-1)} - (k+1)\bar{a}_{k+1} R^{k+1} - \bar{b}_{k-1} R^{k-1} &= U_{-k} \\ -(k-1) a_{-(k-1)} \rho^2 + \bar{a}_{k+1} \rho^{2(k+1)} + b_{-(k+1)} &= 0, \end{aligned} \tag{59}$$

while $p = +k,\ \ k = 2, 3, \dots$ results in

$$\begin{aligned} \kappa a_{k+1} R^{k+1} + (k-1)\bar{a}_{-(k-1)} R^{-(k-1)} - \bar{b}_{-(k+1)} R^{-(k+1)} &= U_k \\ (k+1) a_{k+1} \rho^{2(k+1)} + \bar{a}_{-(k-1)} \rho^2 + b_{k-1} \rho^{2k} &= 0. \end{aligned} \tag{60}$$

These systems may be represented in a recursive form, convenient for numerical computations and further analysis. Namely,

$$S_k(\rho) \cdot \begin{bmatrix} a_{k+1} \\ b_{k-1} \end{bmatrix} = \begin{bmatrix} U_k \\ \bar{U}_{-k} \end{bmatrix} \tag{61}$$

where S_k has entries

$$\begin{aligned} S_k(\rho)_{11} &= \kappa R^{k+1} - (k^2-1) R^{1-k} \rho^{2k} + k^2 R^{-(k+1)} \rho^{2(k+1)} \\ S_k(\rho)_{12} &= -(k-1)(R^{1-k} \rho^{2(k-1)} - R^{-(k+1)} \rho^{2k}) \\ S_k(\rho)_{21} &= -(k+1)(R^{k+1} + \kappa R^{1-k} \rho^{2k}) \\ S_k(\rho)_{22} &= -R^{k-1} - \kappa R^{1-k} \rho^{2(k-1)} \end{aligned} \tag{62}$$

as well as

$$\begin{bmatrix} a_{-(k-1)} \\ b_{-(k+1)} \end{bmatrix} = T_k(\rho) \cdot \begin{bmatrix} \bar{a}_{k+1} \\ \bar{b}_{k-1} \end{bmatrix}, \tag{63}$$

where

$$T_k(\rho) = \begin{bmatrix} -(k+1)\rho^{2k} \ , & -\rho^{2(k-1)} \\ -k^2 \rho^{2(k+1)} \ , & -(k-1)\rho^{2k} \end{bmatrix}. \tag{64}$$

In fact the formulas (63), (61) are correct also for $k = 0, 1$ and in the limit $\rho \longrightarrow 0+$, but the derivation must separate these cases.

Thus, for given $k > 1$ and using some initial a_k, b_k obtained earlier, we may first compute a_{k+1}, b_{k-1} using (61) and then $a_{-(k-1)}, b_{-(k+1)}$ from (63).

We may now use the above results for the asymptotic analysis of the solution. To simplify the formulas, we assume $R = 1$, which means only rescaling and does not diminish generality (in general case ρ would be replaced by ρ/R). Then by direct computation we get the following bounds for the differences between the coefficients on the full circle and the ring. For the initial values of k they read

$$
\begin{aligned}
d_0 - d_0^0 &= -\rho^4 \frac{2}{\kappa(\kappa R^4+\rho^4)} \bar{U}_1 \\
a_1 - a_1^0 &= -\rho^2 \frac{2}{(\kappa-1)R((\kappa-1)R^2+2\rho^2)} \Re U_0 \\
a_2 - a_2^0 &= -\rho^4 \frac{1}{\kappa R^2(\kappa R^4+\rho^4)} U_1
\end{aligned} \tag{65}
$$

and for higher values

$$|a_3 - a_3^0| \le \Lambda \left(|U_2|\rho^4 + |U_{-2}|\rho^2\right) \tag{66}$$

and for $k = 4, 5, \dots$

$$|a_k - a_k^0| \le \Lambda \left(|U_{k-1}|\rho^{3(k-1)/2} + |U_{1-k}|\rho^{3(k-2)/2}\right) \tag{67}$$

where the exponent $k/2$ has been used to counteract the growth of k^2 in terms like $k^2\rho^{k/2}$. Similarly

$$|b_1 - b_1^0| \le \Lambda \left(|U_2|\rho^4 + |U_{-2}|\rho^2\right) \tag{68}$$

and for $k = 2, 3, \dots$

$$|b_k - b_k^0| \le \Lambda \left(|U_{k+1}|\rho^{3(k+1)/2} + |U_{-(k+1)}|\rho^{3k/2}\right). \tag{69}$$

From relation (63) we get further estimates

$$
\begin{aligned}
|a_{-k}| &\le \Lambda \rho^{2k} \left(|U_{k+1}| + |U_{-(k+1)}|\right), \quad k = 1, 2, \dots \\
|b_{-k}| &\le \Lambda \rho^{2(k-1)} \left(|U_{k-1}| + |U_{1-k}|\right), \quad k = 3, 4, \dots
\end{aligned} \tag{70}
$$

Here Λ is a constant independent from ρ and U_i. Observe that the corrections proportional to ρ^2 are present only in a_1, b_1, a_3, b_{-1}, a_{-1}. The rest is of the order at least $O(\rho^3)$ (in fact $O(\rho^4)$).

These estimates may be translated into the following theorem concerning the solution of the elasticity system in the ring.

Theorem 3. *The condition*

$$\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\partial B(R))} \le \Lambda_0 \tag{71}$$

which in terms of U_i means

$$\sum_{k=-\infty}^{k=+\infty} \sqrt{1+k^2}\, |U_k|^2 \le \Lambda_0 \tag{72}$$

ensures that the expression for elastic energy concentrated in the ring splits into the one corresponding to the full circle, correction proportional to ρ^2 and the rest, which is uniformly of the order $\Lambda_0\rho^3$.

4.1 Numerical Illustration

We shall show two solutions corresponding to different boundary conditions on the outer boundary, obtained using the representations derived above in terms of (in these particular cases finite) complex series.
Rugby-like deformation. Let us take $u_r = s_0 \cos^2\theta = \frac{1}{2}s_0 + \frac{1}{2}s_0\cos 2\theta$. Hence

$$[U_k,\ k \in \mathbb{Z}] = [\ldots, \frac{1}{2}\mu s_0, 0,\ U_0 = \mu s_0\ , 0, \frac{1}{2}\mu s_0, \ldots]. \tag{73}$$

The resulting distortion for size of the internal hole $\rho = 0.2$ at the radius $r = 0.3$ are shown in Fig. 1 (solid line - undeformed, dashed - deformed ring, dotted - deformed ball):

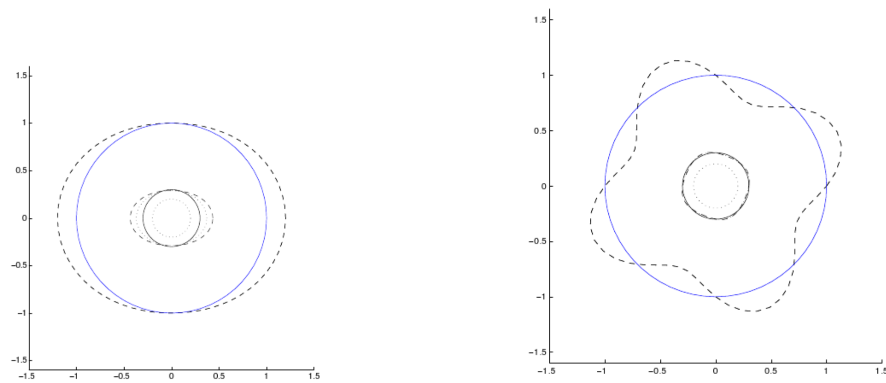

Fig. 1 Rugby-like and bubble-like distortions

Bubble-like deformation. Now we take $u_r = s_0 \sin 4\theta$. Hence

$$[U_k,\ k \in \mathbb{Z}] = [\ldots, \mu s_0 i, 0, 0, 0,\ A_0 = 0\ , 0, 0, 0, -\mu s_0 i, \ldots]. \tag{74}$$

The resulting distortions for $\rho = 0.2$ and $r = 0.3$ shows also Fig. 1, using the same types of lines.

In the second numerical experiment - bubble - only U_{-4} and U_4 were nonzero, which means that the difference between positions of the contour $r = 0.3$ for full circle and the ring should behave like ρ^6. In the first experiment it should be ρ^2, i.e. the influence of boundary condition should vanish quicker. The deformations for $\rho = 0.2$ and several intermediate radii (dashed - undeformed, solid - deformed contours) are visible in Fig. 2 and they confirm this observation.

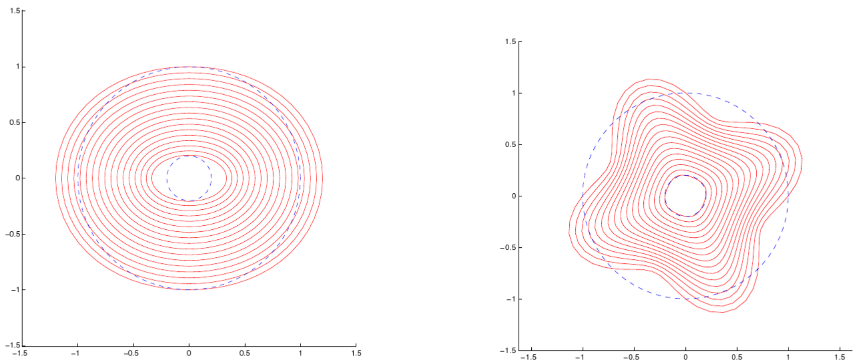

Fig. 2 The pattern of distortions for both experiments

5 Correction Term for Steklov-Poincaré Operator

The elastic energy contained in the ring has the form

$$2\mathscr{E}(\rho,R) = \int_{C(\rho,R)} \sigma(u_\rho) : \varepsilon(u_\rho)\, dx = \int_{\Gamma_R} u_\rho \sigma(u_\rho).n\, ds. \tag{75}$$

Since $u_\rho = u$ on Γ_R,

$$2\mathscr{E}(\rho,R) = \int_{\Gamma_R} u\sigma(u_\rho).n\, ds. \tag{76}$$

Now $\sigma(u_\rho)$ is in fact of the form $\sigma(u_\rho) = \sigma_\rho(u)$, because $u_\rho = u$ on Γ_R, which means that $u_\rho = u_\rho(u)$. If we split σ_ρ into

$$\sigma_\rho(u) = \sigma^0 + \rho^2\sigma^1(u) + O(\rho^4) \tag{77}$$

then

$$2\mathscr{E}(\rho,R) = 2\mathscr{E}(0,R) + \rho^2 \int_{\Gamma_R} u\sigma^1(u).n\, ds + O(\rho^4). \tag{78}$$

Thus finding $\mathscr{A}_1$ reduces to computing $\sigma^1(u)$. From (49), (50) we know that $\sigma_\rho(u)$ is a linear function of infinite vectors $a = [a_k,\ k \in \mathbb{Z}]$, $b = [b_k,\ k \in \mathbb{Z}]$, while $\sigma^0(u)$ is the same function of a^0, b^0. Here a^0, b^0 are computed for $B(R)$, while a, b correspond to $C(\rho,R)$. In order to obtain $\sigma^1(u)$ it is enough to express a, b as

$$a = a^0 + \rho^2 a^1 + O(\rho^4), \quad b = b^0 + \rho^2 b^1 + O(\rho^4) \tag{79}$$

because then

$$\sigma^1(u) = \sigma^1(a^1, b^1). \tag{80}$$

In addition, the only nonzero terms in a^1, b^1 are $a^1_3, a^1_1, a^1_{-1}, b^1_{-1}, b^1_1$.

Taking into account that $A = 0$ in (50) for our problem,

$$\phi = \phi^0 + \rho^2 \phi^1 + O(\rho^4), \quad \psi = \psi^0 + \rho^2 \psi^1 + O(\rho^4) \tag{81}$$

where

$$\phi^1 = a_{-1}^1 \frac{1}{z} + a_1^1 z + a_3^1 z^3, \quad \psi^1 = b_{-1}^1 \frac{1}{z} + b_1^1 z. \tag{82}$$

Using formulas derived in preceding section, we may explicitly compute the coefficients appearing in (82).

$$\begin{aligned} a_{-1}^1 &= -\bar{b}_1^0, \quad a_3^1 = \frac{1}{\kappa R^4} b_1^0, \quad b_1^1 = \frac{3+\kappa^2}{\kappa R^2} b_1^0, \\ a_1^1 &= -\frac{2}{(\kappa-1)R^2} \Re a_1^0, \quad b_{-1}^1 = -2\Re a_1^0. \end{aligned} \tag{83}$$

As is obvious from earlier calculations, only U_0, U_2, U_{-2} will contribute to these corrections, Since

$$U_k = \frac{\mu}{\pi} \int_0^{2\pi} (u_r + i u_\theta) e^{-ik\theta} \, d\theta \tag{84}$$

as well as

$$u_r + i u_\theta = (u_1 + i u_2) e^{-i\theta} \tag{85}$$

then

$$\begin{aligned} U_0 &= \frac{\mu}{\pi} \int_0^{2\pi} (u_1 + i u_2) e^{-i\theta} \, d\theta \\ U_2 &= \frac{\mu}{\pi} \int_0^{2\pi} (u_1 + i u_2) e^{-3i\theta} \, d\theta \\ U_{-2} &= \frac{\mu}{\pi} \int_0^{2\pi} (u_1 + i u_2) e^{+i\theta} \, d\theta. \end{aligned} \tag{86}$$

After collecting all formulas we obtain the final expression

$$\begin{aligned} \int_{\Gamma_R} u^\top \sigma^1(u).n \, ds = \frac{1}{R^2} \Big[& \frac{2(\kappa-2)}{(\kappa-1)^2} (\Re U_0)^2 - (\kappa+1)|U_{-2}|^2 \\ & - \frac{9(\kappa+1)}{\kappa^2} |U_2|^2 - \frac{6(\kappa+1)}{\kappa} \Re(U_2 U_{-2}) \Big]. \end{aligned} \tag{87}$$

From (86) it follows that

$$\begin{aligned} \Re U_0 &= \frac{\mu}{\pi} \int_0^{2\pi} (u_1 \cos\theta + u_2 \sin\theta) \, d\theta \\ U_2 &= \frac{\mu}{\pi} \int_0^{2\pi} (u_1 \cos 3\theta + u_2 \sin 3\theta) \, d\theta + i \frac{\mu}{\pi} \int_0^{2\pi} (u_2 \cos 3\theta - u_1 \sin 3\theta) \, d\theta \\ U_{-2} &= \frac{\mu}{\pi} \int_0^{2\pi} (u_1 \cos\theta - u_2 \sin\theta) \, d\theta + i \frac{\mu}{\pi} \int_0^{2\pi} (u_2 \cos\theta + u_1 \sin\theta) \, d\theta. \end{aligned} \tag{88}$$

Here values of displacements are taken as $u_i(R\cos\theta, R\sin\theta)$. After discretization these integrals constitute weighted sums of values of u_i at certain points on Γ_R. If we assume piecewise linear approximation over triangles, then it is well known that

$$u_i^h(x) = x^\top \begin{bmatrix} x_1^1 & x_2^1 & 1 \\ x_1^2 & x_2^2 & 1 \\ x_1^3 & x_2^3 & 1 \end{bmatrix}^{-1} U_i^h = x^\top M^{-1} U_i^h \tag{89}$$

and

$$x^\top M^{-1} U_i^h = (M^{-\top} x)^\top U_i^h = c^\top U_i^h \tag{90}$$

where $u_i^h(x)$ is a value of the approximation of u_i at a point x inside the triangle defined by vertices x^1, x^2, x^3 and U_i^h is a vector of the values of u_i^h at these vertices. Observe that c is a vector of weights with which nodal values enter into the expression for $u_i^h(x)$.

Let now $U^h = [u_1^{h1}, u_2^{h1}, \ldots, u_1^{hK}, u_2^{hK}]^\top$ be a vector of nodal values of u^h for the global triangulation. Then we may write down the following formulae

$$\begin{aligned}
&\frac{\mu}{\pi}\int_0^{2\pi} u_1 \cos\theta \, d\theta = c_{11}^\top U^h && \frac{\mu}{\pi}\int_0^{2\pi} u_2 \sin\theta \, d\theta = s_{21}^\top U^h \\
&\frac{\mu}{\pi}\int_0^{2\pi} u_1 \cos 3\theta \, d\theta = c_{13}^\top U^h && \frac{\mu}{\pi}\int_0^{2\pi} u_2 \sin 3\theta \, d\theta = s_{23}^\top U^h \\
&\frac{\mu}{\pi}\int_0^{2\pi} u_1 \sin\theta \, d\theta = s_{11}^\top U^h && \frac{\mu}{\pi}\int_0^{2\pi} u_2 \cos\theta \, d\theta = c_{21}^\top U^h \\
&\frac{\mu}{\pi}\int_0^{2\pi} u_1 \sin 3\theta \, d\theta = s_{13}^\top U^h && \frac{\mu}{\pi}\int_0^{2\pi} u_2 \cos 3\theta \, d\theta = c_{23}^\top U^h .
\end{aligned} \tag{91}$$

Here s_{ij}, c_{ij} are sparse vectors of weights with which nodal values of u enter into appropriate integrals. In this notation

$$\begin{aligned}
(\Re U_0)^2 &= \|(c_{11}+s_{21})^\top U^h\|^2 \\
|U_2|^2 &= \|(c_{13}+s_{23})^\top U^h\|^2 + \|(c_{23}-s_{13})^\top U^h\|^2 \\
|U_{-2}|^2 &= \|(c_{11}-s_{21})^\top U^h\|^2 + \|(c_{21}+s_{11})^\top U^h\|^2 \\
\Re(U_2 U_{-2}) &= (U^h)^\top (c_{13}+s_{23})(c_{11}-s_{21}) U^h \\
&\quad - (U^h)^\top (c_{23}-s_{13})(c_{21}+s_{11}) U^h .
\end{aligned} \tag{92}$$

Taking into account (87) we may conclude that the first term in the correction of energy is a well defined quadratic form. Similar, only more complicated expressions may be obtained for further asymptotics corresponding to ρ^4.

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