Shape Sensitivity Analysis for Compressible Navier-Stokes Equations

Pavel I. Plotnikov, Evgenya V. Ruban, and Jan Sokołowski

Abstract In the paper compressible, stationary Navier-Stokes (N-S) equations are considered. The model is well-posed, there exist weak solutions in bounded domains, subject to inhomogeneous boundary conditions. The shape sensitivity analysis is performed for N-S boundary value problems, in the framework of small perturbations of the so-called *approximate solutions*. The approximate solutions are determined from Stokes problem and the small perturbations are given by solutions to the full nonlinear model. Such solutions are unique. The differentiability of the specific solutions with respect to the coefficients of differential operators implies the shape differentiability of the drag functional. The shape gradient of the drag functional is derived in the classical and useful for computations form, an appropriate adjoint state is introduced to this end. The proposed method of shape sensitivity analysis is general, and can be used to establish the well-posedness for distributed and boundary control problems as well as for inverse problems in the case of the state equations in the form of compressible Navier-Stokes equations.

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1 Introduction

Shape optimization for compressible Navier-Stokes equations (N-S) is important for applications [9] and it is investigated from numerical point of view in the field of scientific computations, however the mathematical analysis of such problems is not available in the existing literature. One of the reasons is the lack of the existence results for inhomogeneous boundary value problems for such equations. We refer the reader to the chapter [21] for the state of art and some new results in this domain.

The results established in the paper lead in particular to the first order optimality conditions for a class of shape optimization problems for compressible Navier-Stokes equations.

Our results for Fourier-Navier-Stokes (F-N-S) and N-S boundary value problems can be presented according to the following plan.

- *Mathematical modeling, well posedness of solutions to the boundary value problems.* The most general setting for such analysis is introduced in [20] and covers the F-N-S boundary value problems in bounded domains with inhomogeneous boundary conditions. We point out that in [18] the diatomic gases are considered and the existence of solutions for the mathematical models is shown. The shape differentiability of solutions is proved in [19] for the Navier-Stokes boundary value problems in bounded domains with inhomogeneous boundary conditions.
- *The drag functional* is minimized, however the same approach can be used for more general problems of shape optimization including the lift maximization and the density distribution optimization at the outlet of the flow domain.
- *Framework for the shape sensitivity analysis.* The new results are derived for small perturbations of the approximate solutions to compressible N-S equations. In [19] the shape sensitivity analysis is performed with respect to the adjugate matrix defined for the Jacobi matrix of a given domain transformation mapping. Our approach allows for substantial simplification of the sensitivity analysis compared to the existing results obtained in the case of incompressible fluids by using the velocity or perturbation of identity methods of shape sensitivity analysis.
- *Material derivatives* of solutions to compressible N-S equations in the fixed domain setting are obtained in [19]. The shape differentiability of solutions for compressible N-S boundary value problems is shown with respect to weak norms, i.e., in the negative Sobolev spaces for the hyperbolic component, that is, the transport equation, however the obtained material derivatives are sufficiently regular in order to obtain the shape gradients given by some functions, and such a result is actually very useful for possible application of numerical methods of shape optimization of the level set type – since the shape gradients are the coefficients of the non linear hyperbolic equation.
- *Shape gradient* of the drag functional is determined by means of the complicated adjoint state, and we observe that the expression obtained is sufficiently smooth and given by a function, it implies that e.g., the level set method can be employed for numerical solution of the shape optimization for the drag minimization.

The shape optimization for compressible Navier-Stokes equations is an important branch of the research, e.g. in aerodynamics. The main difficulty in analysis of such optimization problems is the mathematical modeling, i.e., the lack of the existence results for inhomogeneous boundary value problems in bounded domains [18]. The authors proved the existence of an optimal shape for drag minimization in three spatial dimensions under the Mosco convergence of admissible domains and assuming that the family of admissible domains is nonempty [17]. This is a result on the compactness of the set of solutions to Navier-Stokes equations for the admissible family of obstacles, we refer the reader to [14]-[17] for further details. The shape differentiability of solutions to N-S equations with respect to boundary perturbations is shown in [19], and leads to the optimality system for the shape optimization problem under considerations.

2 Compressible, Stationary Fourier-Navier-Stokes Equations

The most general results on the existence of solutions to N-S equations from the point of view of drag minimization are established in [20] for the complete model including the heat conduction. The model is formulated in the following way.

Modeling. Let us consider the following set of equations with the state variables: $\mathbf{u}$ is the velocity field in the bounded domain Ω in three spatial dimensions, $\rho > 0$ stands for the mass density, and $\vartheta \geq 0$ is the temperature

$$\Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u} = k \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \omega \nabla\big(\rho(1+\vartheta)\big) + \rho\, \mathbf{g} \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div}(\rho \mathbf{u}) = 0 \quad \text{in } \Omega, \tag{2}$$

$$\Delta \vartheta = k\gamma^{-1}\Big(\rho \mathbf{u} \nabla \vartheta + (\gamma - 1)(1+\vartheta)\rho \operatorname{div} \mathbf{u}\Big) - k\omega^{-1}(1-\gamma^{-1})D \quad \text{in } \Omega, \tag{3}$$

where $k = R$, $\omega = R/(\gamma\varepsilon^2)$, R is the Reynolds number, ε is the Mach number, λ is the viscosity ratio, γ is the adiabatic constant, $\mathbf{g}$ denotes the dimensionless mass force, and the dissipative function D is defined by the equality

$$D = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^*)^2 + (\lambda - 1)\operatorname{div} \mathbf{u}^2. \tag{4}$$

The governing equations should be supplemented with the boundary conditions. The velocity of the gas coincides with a given vector field $\mathbf{U} \in C^\infty(\mathbb{R}^3)^3$ on the surface $\partial\Omega$. In this framework, the boundary of the flow domain is divided into three subsets: the inlet Σ_{in}, outgoing set Σ_{out}, and characteristic set Σ_0 defined by the equalities

$$\Sigma_{\text{in}} = \{x \in \overline{\Sigma} : \mathbf{U} \cdot \mathbf{n} < 0\}, \quad \Sigma_{\text{out}} = \{x \in \overline{\Sigma} : \mathbf{U} \cdot \mathbf{n} > 0\}, \tag{5}$$

$\Sigma_0 = \{x \in \partial\Omega : \mathbf{U} \cdot \mathbf{n} = 0\}$, where $\mathbf{n}$ stands for the unit outward normal to $\partial\Omega$.

The state variables satisfy the boundary conditions

$$\mathbf{u} = \mathbf{U}, \quad \vartheta = 0 \text{ on } \partial\Omega, \quad \rho = g \text{ on } \Sigma_{\text{in}}, \tag{6}$$

in which g is a given positive function.

Emergent vector field conditions. The existence of solutions to the mathematical model can be established under geometrical conditions related to the characteristic set $\Gamma \subset \Sigma_0$ and the boundary data $\mathbf{U}$ for the velocity field, see Fig. 1 for a specific geometry of the bounded flow domain Ω with an obstacle S. Note that the boundary of the obstacle is not important for such a condition. The emergent vector field condition is known in the theory of PDE's for the oblique derivative problems, and it is introduced and exploited in our papers for the compressible N-S equations with the hyperbolic component for the mass transport. In our case the condition allows us to construct in an appropriate way the solutions to the mass transport equation.

Assume that a characteristic set $\Gamma \subset \partial\Omega$ and a given vector field $\mathbf{U}$ satisfy the following conditions, referred to as the *emergent vector field conditions*.

Emergent vector field conditions: The set Γ is a a closed C^∞ one-dimensional manifold. Moreover, there is a positive constant c such that

$$\mathbf{U} \cdot \nabla(\mathbf{U} \cdot \mathbf{n}) > c > 0 \text{ on } \Gamma. \tag{7}$$

Since the vector field $\mathbf{U}$ is tangent to $\partial\Omega$ on Γ, the quantity in the left-hand side of (7) is well defined.

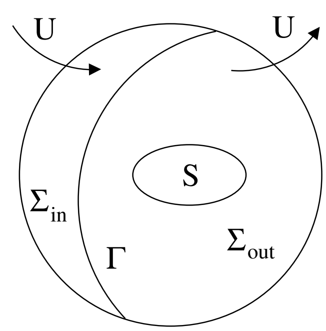

Fig. 1 Characteristic set $\Gamma \subset \partial B$ on the exterior boundary of the flow domain Ω

This condition is obviously fulfilled for all strictly convex domains and constant vector fields. It has a simple geometric interpretation, that $\mathbf{U} \cdot \mathbf{n}$ only vanishes up to the first order at Γ, and for each point $P \in \Gamma$, the vector $\mathbf{U}(P)$ points to the part of $\partial\Omega$ where $\mathbf{U}$ is an exterior vector field.

Boundary value problem. We use the approximate solutions of the F-N-S boundary value problems in order to show the existence, uniqueness and the stability of solutions to F-N-S boundary value problems. The method is general and well suited in the framework of mathematical modeling in the shape optimization, in the optimal control and in solution of inverse problems.

Let us consider the following boundary value problem for the incompressible Navier-Stokes equations

$$\Delta \mathbf{u}_0 - \nabla p_0 = k \mathrm{div}\,(\mathbf{u}_0 \otimes \mathbf{u}_0), \quad \mathrm{div}\,\mathbf{u}_0 = 0 \text{ in } \Omega, \quad \mathbf{u}_0 = \mathbf{U} \text{ on } \partial\Omega, \quad \Pi p_0 = p_0. \tag{8}$$

In our notations Π is the projection,

$$\Pi u = u - \frac{1}{\mathrm{meas}\,\Omega} \int_{\Omega} u\,dx. \tag{9}$$

It is well known that for each $\mathbf{U} \in C^\infty(\Omega)$ satisfying the orthogonality conditions

$$\int_{\partial\Omega} \mathbf{U} \cdot \mathbf{n}\,ds = 0 \tag{10}$$

and all sufficiently small k, this problem has a unique C^∞-solution. The triple $(\rho_0, \mathbf{u}_0, \vartheta_0) =: (1, \mathbf{u}_0, 0)$ is an approximate solution for small Mach numbers.

Theorem 1. *There exist positive constants k^*, ω^* such that for each fixed $k \in [0, k^*]$, and all $\omega > \omega^*$, the stationary N-S-F problem has a solution $(\mathbf{u}_\omega, \rho_\omega, \vartheta_\omega) \in (X^{1+s,r})^3 \times X^{s,r} \times X^{1+s,r}$ such that*

$$\|\mathbf{u}_\omega - \mathbf{u}_0\|_{1+s,r} + \|\rho_\omega - 1\|_{s,r} + \|\vartheta_\omega\|_{1+s,r} \to 0 \;\; as \;\; \omega \to \infty. \tag{11}$$

Proof is given in [20]. Applications of Theorem 1 to the existence of optimal shapes, as well as to the shape differentiability of solutions to the F-N-S boundary value problem are presented in the forthcoming publications.

3 Compressible, Stationary Navier-Stokes Equations

We restrict ourselves to the inhomogeneous boundary value problems for compressible, stationary Navier-Stokes equations. Such modeling is considered in [14]-[19]. In particular, the well-posedness for inhomogeneous boundary value problems of elliptic-hyperbolic type is shown in [19]. Analysis is performed for small perturbations of the approximate solutions, which are determined from the Stokes problem. The existence and uniqueness of solutions close to approximate solution are proved, and in addition, the differentiability of solutions with respect to the coefficients of differential operators is shown in [19]. The results on the well-posedness of nonlinear problem are interesting on their own, and are used to obtain the shape differentiability of the drag functional for incompressible Navier-Stokes equations. The shape gradient of the drag functional is derived in the classical and useful for computations form, an appropriate adjoint state is introduced to this end. The shape derivatives of solutions to the Navier-Stokes equations are given by smooth functions, however the shape differentiability is shown in a weak norm. The method of analysis proposed in [19] is general, and can be used to establish the well-posedness for distributed and boundary control problems as well as for inverse problems in the case of the state equations in the form of compressible Navier-Stokes equations. The differentiability of solutions to the Navier-Stokes equations with respect to the data leads to the first order necessary conditions for a broad class of optimization problems.

4 Drag Minimization

We present an example of shape optimization in aerodynamics. Mathematical analysis of the drag minimization problem for compressible Navier-Stokes equations can be found, e.g., in [17] on the domain continuity of solutions, and in [19] on the shape differentiability of the drag functional.

Mathematical model in the form of N-S equations. We assume that the viscous gas occupies the double-connected domain $\Omega = B\backslash S$, where $B \subset \mathbb{R}^3$ is a hold-all domain with the smooth boundary $\Sigma = \partial B$, and $S \subset B$ is a compact obstacle. Furthermore, we assume that the velocity of the gas coincides with a given vector field $\mathbf{U} \in C^\infty(\mathbb{R}^3)^3$ on the surface Σ. In this framework, the boundary of the flow domain Ω is divided into the three subsets (see (5)), inlet Σ_{in}, outgoing set Σ_{out}, and Σ_0. In its turn, the compact $\Gamma = \Sigma_0 \cap \Sigma$ splits the surface Σ into three disjoint parts $\Sigma = \Sigma_{\text{in}} \cup \Sigma_{\text{out}} \cup \Gamma$. The problem is to find the velocity field $\mathbf{u}$ and the gas density ρ satisfying the following equations along with the boundary conditions

$$\Delta \mathbf{u} + \lambda \nabla \text{div} \mathbf{u} = R\rho \mathbf{u} \cdot \nabla \mathbf{u} + \frac{R}{\varepsilon^2} \nabla p(\rho) \text{ in } \Omega, \quad \text{div}\,(\rho \mathbf{u}) = 0 \text{ in } \Omega, \tag{12}$$

$$\mathbf{u} = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u} = 0 \text{ on } \partial S, \quad \rho = \rho_0 \text{ on } \Sigma_{\text{in}}, \tag{13}$$

where the pressure $p = p(\rho)$ is a smooth, strictly monotone function of the density, ε is the Mach number, R is the Reynolds number, λ is the viscosity ratio, and ρ_0 is a positive constant.

Drag minimization. One of the main applications of the theory of compressible viscous flows is the optimal shape design in aerodynamics. The classical example is the problem of minimization of the drag of airfoil traveling in the atmosphere with uniform speed $\mathbf{U}_\infty$. Recall that in our framework the hydrodynamical force acting on the body S is defined by the formula,

$$\mathbf{J}(S) = -\int_{\partial S} (\nabla \mathbf{u} + (\nabla \mathbf{u})^* + (\lambda - 1)\text{div}\, \mathbf{u}\mathbf{I} - \frac{R}{\varepsilon^2} p\mathbf{I}) \cdot \mathbf{n} dS. \tag{14}$$

In a frame attached to the moving body the drag is the component of $\mathbf{J}$ parallel to $\mathbf{U}_\infty$,

$$J_D(S) = \mathbf{U}_\infty \cdot \mathbf{J}(S), \tag{15}$$

and the lift is the component of $\mathbf{J}$ in the direction orthogonal to $\mathbf{U}_\infty$. For the fixed data, the drag can be regarded as a functional depending on the shape of the obstacle S. The minimization of the drag and the maximization of the lift are among shape optimization problems of some practical importance. We show the shape differentiability of the drag functional with respect to the boundary variations.

5 Shape Sensitivity Analysis

We start with description of our framework for shape sensitivity analysis, or more general, for well-posedness of compressible N-S equations. The detailed proofs of the results presented in the section are given in the forthcoming paper [19]. To this end we choose the vector field $\mathbf{T} \in C^2(\mathbb{R}^3)^3$ vanishing in the vicinity of Σ, and define the mapping

$$y = x + \varepsilon \mathbf{T}(x), \tag{16}$$

which describes the perturbation of the shape of the obstacle. We refer the reader to [25] for more general framework and results in shape optimization. For small ε, the mapping $x \mapsto y$ takes diffeomorphically the flow region Ω onto $\Omega_\varepsilon = B \setminus S_\varepsilon$, where the perturbed obstacle $S_\varepsilon = y(S)$. Let $(\bar{\mathbf{u}}_\varepsilon, \bar{\rho}_\varepsilon)$ be solutions to problem (12) in Ω_ε. After substituting $(\bar{\mathbf{u}}_\varepsilon, \bar{\rho}_\varepsilon)$ into the formulae for $\mathbf{J}$, the drag becomes the function of the parameter ε. Our aim is, in fact, to prove that this function is well-defined and differentiable at $\varepsilon = 0$. This leads to the first order shape sensitivity analysis for solutions to compressible Navier-Stokes equations. It is convenient to reduce such an analysis to the analysis of dependence of solutions with respect to the coefficients of the governing equations. To this end, we introduce the functions $\mathbf{u}_\varepsilon(x)$ and $\rho_\varepsilon(x)$ defined in the unperturbed domain Ω by the formulae

$$\mathbf{u}_\varepsilon(x) = \mathbf{N}\bar{\mathbf{u}}_\varepsilon(x + \varepsilon \mathbf{T}(x)), \quad \rho_\varepsilon(x) = \bar{\rho}_\varepsilon(x + \varepsilon \mathbf{T}(x)), \tag{17}$$

where

$$\mathbf{N}(x) = [\det(\mathbf{I} + \varepsilon \mathbf{T}'(x))(\mathbf{I} + \varepsilon \mathbf{T}'(x))]^{-1}. \tag{18}$$

is the adjugate matrix of the Jacobi matrix $\mathbf{I} + \varepsilon \mathbf{T}'$. Furthermore, we also use the notation $\mathfrak{g}(x) = \sqrt{\det \mathbf{N}}$. It is easy to see that the matrix $\mathbf{N}(x)$ depends analytically upon the small parameter ε and

$$\mathbf{N} = \mathbf{I} + \varepsilon \mathbf{D}(x) + \varepsilon^2 \mathbf{D}_1(\varepsilon, x), \tag{19}$$

where $\mathbf{D} = \operatorname{div} \mathbf{T}\mathbf{I} - \mathbf{T}'$. Calculations show that for $\mathbf{u}_\varepsilon, \rho_\varepsilon$, the following boundary value problem is obtained

$$\Delta \mathbf{u}_\varepsilon + \nabla\Big(\lambda \mathfrak{g}^{-1} \operatorname{div} \mathbf{u}_\varepsilon - \frac{R}{\varepsilon^2} p(\rho_\varepsilon)\Big) = \mathscr{A}\mathbf{u}_\varepsilon + R\mathscr{B}(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \text{ in } \Omega, \tag{20}$$

$$\operatorname{div}\big(\rho_\varepsilon \mathbf{u}_\varepsilon\big) = 0 \text{ in } \Omega, \tag{21}$$

$$\mathbf{u}_\varepsilon = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u}_\varepsilon = 0 \text{ on } \partial S, \tag{22}$$

$$\rho_\varepsilon = \rho_0 \text{ on } \Sigma_{\text{in}}. \tag{23}$$

Here, the linear operator $\mathscr{A}$ and the nonlinear mapping $\mathscr{B}$ are defined in terms of $\mathbf{N}$,

$$\mathscr{A}(\mathbf{u}) = \Delta\mathbf{u} - \mathbf{N}^{-1}\mathrm{div}\,\big(\mathfrak{g}^{-1}\mathbf{N}\mathbf{N}^*\nabla(\mathbf{N}^{-1}\mathbf{u})\big), \tag{24}$$

$$\mathscr{B}(\rho,\mathbf{u},\mathbf{w}) = \rho(\mathbf{N}^*)^{-1}\Big(\mathbf{u}\nabla\big(\mathbf{N}^{-1}\mathbf{w}\big)\Big). \tag{25}$$

The specific structure of the matrix $\mathbf{N}$ does not play any particular role in the further analysis. Therefore, we consider a general problem of the existence, uniqueness and dependence on coefficients of the solutions to equations (20)-(23) under the assumption that $\mathbf{N}$ is a given matrix-valued function which is close, in an appropriate norm, to the identity mapping $\mathbf{I}$ and coincides with $\mathbf{I}$ in the vicinity of Σ. By abuse of notations, we write simply $\mathbf{u}$ and ρ instead of $\mathbf{u}_\varepsilon$ and ρ_ε, when studying the well-posedness and dependence on $\mathbf{N}$. Before formulation of main results we write the governing equation in more transparent form using the change of unknown functions proposed, e.g., in [13]. To do so we introduce *the effective viscous pressure*

$$q = \frac{R}{\varepsilon^2}\,p(\rho) - \lambda\,\mathfrak{g}^{-1}\mathrm{div}\,\mathbf{u}, \tag{26}$$

and rewrite equations (20)-(23) in the equivalent form

$$\Delta\mathbf{u} - \nabla q = \mathscr{A}(\mathbf{u}) + R\mathscr{B}(\rho,\mathbf{u},\mathbf{u}) \quad \text{in } \Omega, \tag{27}$$

$$\mathrm{div}\,\mathbf{u} = a\sigma_0 p(\rho) - \frac{\mathfrak{g}q}{\lambda} \quad \text{in } \Omega, \tag{28}$$

$$\mathbf{u}\cdot\nabla\rho + \mathfrak{g}\sigma_0 p(\rho)\,\rho = \frac{\mathfrak{g}q}{\lambda}\rho \quad \text{in } \Omega, \tag{29}$$

$$\mathbf{u} = \mathbf{U} \quad \text{on } \Sigma, \quad \mathbf{u} = 0 \quad \text{on } \partial S, \tag{30}$$

$$\rho = \rho_0 \quad \text{on } \Sigma_{\mathrm{in}}, \tag{31}$$

where $\sigma_0 = R/(\lambda\varepsilon^2)$. We point out that the solutions to the compressible N-S equations are determined in the form (34). In the new variables $(\mathbf{u}, q, \rho)$ the expression for the force $\mathbf{J}$ reads

$$\mathbf{J} = -\int_\Omega \big[\mathfrak{g}^{-1}\big(\mathbf{N}^*\nabla(\mathbf{N}\mathbf{u}) + \nabla(\mathbf{N}\mathbf{u})^*\mathbf{N} - \mathrm{div}\,\mathbf{u}\big) - q - R\rho\,\mathbf{u}\otimes\mathbf{u}\big]\mathbf{N}^*\nabla\eta\,dx, \tag{32}$$

where $\eta \in C^\infty(\Omega)$ is an arbitrary function, which is equal to 1 in an open neighborhood of the obstacle S and 0 in a vicinity of Σ. The value of $\mathbf{J}$ is independent of the choice of the function η.

6 Perturbations of the Approximate Solutions

We assume that $\lambda \gg 1$ and $R \ll 1$, which corresponds to almost incompressible flow with low Reynolds number. In such a case, the *approximate solutions* to problem

(27)-(31) can be chosen in the form $(\rho_0, \mathbf{u}_0, q_0)$, where ρ_0 is a constant in boundary condition (31), and $(\mathbf{u}_0, q_0)$ is a solution to the boundary value problem for the Stokes equations,

$$\Delta \mathbf{u}_0 - \nabla q_0 = 0, \ \operatorname{div} \mathbf{u}_0 = 0 \text{ in } \Omega, \ \mathbf{u}_0 = \mathbf{U} \text{ on } \Sigma, \ \mathbf{u}_0 = 0 \text{ on } \partial S, \ \Pi q_0 = q_0 \ , \tag{33}$$

where Π is the projector (9). Equations (33) can be obtained as the limit of equations (27)-(31) for the passage $\lambda \to \infty$, $R \to 0$. It follows from the standard elliptic theory that for the boundary $\partial\Omega \in C^\infty$, we have $(\mathbf{u}_0, q_0) \in C^\infty(\Omega)$. We look for solutions to problem (27)-(31) in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \quad \rho = \rho_0 + \varphi, \quad q = q_0 + \lambda \sigma_0 p(\rho_0) + \pi + \lambda m, \tag{34}$$

with the unknown functions $\vartheta = (\mathbf{v}, \pi, \varphi)$ and the unknown constant m. Substituting (34) into (27)-(31) we obtain the following boundary problem for ϑ,

$$\Delta \mathbf{v} - \nabla \pi = \mathscr{A}(\mathbf{u}) + R\mathscr{B}(\rho, \mathbf{u}, \mathbf{u}) \text{ in } \Omega, \tag{35}$$

$$\operatorname{div} \mathbf{v} = \mathfrak{g}\Big(\frac{\sigma}{\rho_0}\varphi - \Psi[\vartheta] - m\Big) \text{ in } \Omega, \tag{36}$$

$$\mathbf{u} \cdot \nabla \varphi + \sigma \varphi = \Psi_1[\vartheta] + m\mathfrak{g}\rho \text{ in } \Omega, \tag{37}$$

$$\mathbf{v} = 0 \text{ on } \partial\Omega, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}}, \quad \Pi\pi = \pi, \tag{38}$$

where

$$\Psi_1[\vartheta] = \mathfrak{g}\Big(\rho\Psi[\vartheta] - \frac{\sigma}{\rho_0}\varphi^2\Big) + \sigma\varphi(1 - \mathfrak{g}), \tag{39}$$

$$\Psi[\vartheta] = \frac{q_0 + \pi}{\lambda} - \frac{\sigma}{p'(\rho_0)\rho_0} H(\varphi), \tag{40}$$

$$\sigma = \sigma_0 p'(\rho_0)\rho_0, \tag{41}$$

$$H(\varphi) = p(\rho_0 + \varphi) - p(\rho_0) - p'(\rho_0)\varphi, \tag{42}$$

the vector field $\mathbf{u}$ and the function ρ are given by (34). Finally, we specify the constant m. In our framework, in contrast to the case of homogeneous boundary problem, the solution to such a problem is not trivial. Note that, since $\operatorname{div} \mathbf{v}$ is of the null mean value, the right-hand side of equation (37) must satisfy the compatibility condition

$$m \int_\Omega \mathfrak{g}\, dx = \int_\Omega \mathfrak{g}\big(\frac{\sigma}{\rho_0}\varphi - \Psi[\vartheta]\big)\, dx, \tag{43}$$

which formally determines m. This choice of m leads to essential mathematical difficulties. To make this issue clear note that in the simplest case $\mathfrak{g} = 1$ we have $m = \rho_0^{-1}\sigma(\mathbf{I} - \Pi)\varphi + O(|\vartheta|^2, \lambda^{-1})$, and the principal linear part of the governing equations (35)-(38) becomes

$$\begin{pmatrix} \Delta & -\nabla & 0 \\ \operatorname{div} & 0 & -\frac{\sigma}{\rho_0} \\ 0 & 0 & \mathbf{u}\nabla + \sigma \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \pi \\ \varphi \end{pmatrix} + \begin{pmatrix} 0 \\ m \\ -m\rho_0 \end{pmatrix} \sim \begin{pmatrix} \Delta \mathbf{v} - \nabla \pi \\ \operatorname{div} \mathbf{v} - \frac{\sigma}{\rho_0} \Pi \varphi \\ \mathbf{u}\nabla\varphi + \sigma \Pi \varphi \end{pmatrix} \tag{44}$$

Hence, the question of solvability of the linearized equations derived for (35)-(38), (46), (47) can be reduced to the question of solvability of the boundary value problem for nonlocal transport equation

$$\mathbf{u}\nabla\varphi + \sigma\Pi\varphi = f, \tag{45}$$

which is very difficult because of the loss of maximum principle. In fact, this question is concerned with the problem of the control of the total gas mass in compressible flows. Recall that the absence of the mass control is the main obstacle for proving the global solvability of inhomogeneous boundary problems for compressible Navier-Stokes equations, we refer to [8] for discussion. In order to cope with this difficulty we write the compatibility condition in a sophisticated form, which allows us to control the total mass of the gas. To this end we introduce the auxiliary function ζ satisfying the equations

$$-\operatorname{div}(\mathbf{u}\zeta) + \sigma\zeta = \sigma\mathfrak{g} \text{ in } \Omega, \quad \zeta = 0 \text{ on } \Sigma_{\text{out}}, \tag{46}$$

and fix the constant m as follows

$$m = \varkappa \int_\Omega (\rho_0^{-1}\Psi_1[\vartheta]\zeta - \mathfrak{g}\Psi[\vartheta])\,dx, \quad \varkappa = \Big(\int_\Omega \mathfrak{g}(1 - \zeta - \rho_0^{-1}\zeta\varphi)\,dx\Big)^{-1}. \tag{47}$$

In this way the auxiliary function ζ becomes an integral part of the solution to problem (35)-(38), (46), (47).

7 Function Spaces

In this section we assemble some technical results which are used throughout the paper. Function spaces play a central role, and we recall some notations, fundamental definitions and properties, which are classical. The proofs of some results given here can be found, e.g. in [19].

For our applications we need the results in three spatial dimensions, however the results are presented for the dimension $d \geq 2$.

Let Ω be the whole space $\mathbb{R}^d$ or a bounded domain in $\mathbb{R}^d$ with the boundary $\partial\Omega$ of class C^1. For an integer $l \geq 0$ and for an exponent $r \in [1, \infty)$, we denote by $H^{l,r}(\Omega)$ the Sobolev space endowed with the norm $\|u\|_{H^{l,r}(\Omega)} = \sup_{|\alpha| \leq l} \|\partial^\alpha u\|_{L^r(\Omega)}$. For real $0 < s < 1$, the fractional Sobolev space $H^{s,r}(\Omega)$ is obtained by the interpolation between $L^r(\Omega)$ and $H^{1,r}(\Omega)$, and consists of all measurable functions with the finite norm

$$\|u\|_{H^{s,r}(\Omega)} = \|u\|_{L^r(\Omega)} + |u|_{s,r,\Omega} \tag{48}$$

where

$$|u|^r_{s,r,\Omega} = \int\limits_{\Omega\times\Omega} |x-y|^{-d-rs}|u(x)-u(y)|^r\,dxdy. \tag{49}$$

In the general case, the Sobolev space $H^{l+s,r}(\Omega)$ is defined as the space of measurable functions with the finite norm $\|u\|_{H^{l+s,r}(\Omega)} = \sup_{|\alpha|\le l}\|\partial^\alpha u\|_{H^{s,r}(\Omega)}$. For $0 < s < 1$, the Sobolev space $H^{s,r}(\Omega)$ is, in fact the interpolation space $[L^r(\Omega), H^{1,r}(\Omega)]_{s,r}$.

Furthermore, the notation $H^{l,r}_0(\Omega)$, with an integer l, stands for the closed subspace of the space $H^{l,r}(\Omega)$ of all functions $u \in L^r(\Omega)$ which being extended by zero outside of Ω belong to $H^{l,r}(\mathbb{R}^d)$.

Denote by $\mathscr{H}^{0,r}_0(\Omega)$ and $\mathscr{H}^{1,r}_0(\Omega)$ the subspaces of $L^r(\mathbb{R}^d)$ and $H^{1,r}(\mathbb{R}^d)$, respectively, of all functions vanishing outside of Ω. Obviously $\mathscr{H}^{1,r}_0(\Omega)$ and $H^{1,r}_0(\Omega)$ are isomorphic topologically and algebraically and we can identify them. However, we need the interpolation spaces $\mathscr{H}^{s,r}_0(\Omega)$ for non-integers, in particular for $s = 1/r$.

Definition 1. For all $0 < s \le 1$ and $1 < r < \infty$, we denote by $\mathscr{H}^{s,r}_0(\Omega)$ the interpolation space $[\mathscr{H}^{0,r}_0(\Omega), \mathscr{H}^{1,r}_0(\Omega)]_{s,r}$ endowed with one of two equivalent norms [19] defined by interpolation method.

It follows from the definition of interpolation spaces that $\mathscr{H}^{s,r}_0(\Omega) \subset H^{s,r}(\mathbb{R}^d)$ and for all $u \in \mathscr{H}^{s,r}_0(\Omega)$,

$$\|u\|_{H^{s,r}(\mathbb{R}^d)} \le c(r,s)\|u\|_{\mathscr{H}^{s,r}_0(\Omega)}, \quad u = 0 \text{ outside } \Omega. \tag{50}$$

In other words, $\mathscr{H}^{s,r}_0(\Omega)$ consists of all elements $u \in H^{s,r}(\Omega)$ such that the extension $\overline{u}$ of u by 0 outside of Ω has the finite $[\mathscr{H}^{0,r}_0(\Omega), \mathscr{H}^{1,r}_0(\Omega)]_{s,r}$-norm. We identify u and $\overline{u}$ for the elements $u \in \mathscr{H}^{s,r}_0(\Omega)$. With this identification it follows that $H^{1,r}_0(\Omega) \subset \mathscr{H}^{s,r}_0(\Omega)$ and the space $C^\infty_0(\Omega)$ is dense in $\mathscr{H}^{s,r}_0(\Omega)$. It is worthy to note that for $0 < s < 1$ and for $1 < r < \infty$, the function $\overline{u}$ belongs to the space $H^{s,r}(\mathbb{R}^d)$ if and only if $u \in H^{s,r}(\Omega)$ and dist $(x, \partial\Omega)^{-s}u \in L^r(\Omega)$. We also point out that the interpolation space $\mathscr{H}^{s,r}_0(\Omega)$ coincides with the Sobolev space $H^{s,r}_0(\Omega)$ for $s \ne 1/r$. Recall that the standard space $H^{s,r}_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ in the $H^{s,r}(\Omega)$-norm.

Embedding theorems. For $sr > d$ and $0 \le \alpha < s - r/d$, the embedding $H^{s,r}(\Omega) \hookrightarrow C^\alpha(\Omega)$ is continuous and compact. In particular, for $sr > d$, the Sobolev space $H^{s,r}(\Omega)$ is a commutative Banach algebra, i.e. for all $u, v \in H^{s,r}(\Omega)$,

$$\|uv\|_{H^{s,r}(\Omega)} \le c(r,s)\|u\|_{H^{s,r}(\Omega)}\|v\|_{H^{s,r}(\Omega)}. \tag{51}$$

If $sr < d$ and $t^{-1} = r^{-1} - d^{-1}s$, then the embedding $H^{s,r}(\Omega) \hookrightarrow L^t(\Omega)$ is continuous. In particular, for $\alpha \le s$, $(s-\alpha)r < d$ and $\beta^{-1} = r^{-1} - d^{-1}(s-\alpha)$,

$$\|u\|_{H^{\alpha,\beta}(\Omega)} \le c(r,s,\alpha,\beta,\Omega)\|u\|_{H^{s,r}(\Omega)}. \tag{52}$$

It follows from (50) that all the embedding inequalities remain true for the elements of the interpolation space $\mathscr{H}_0^{s,r}(\Omega)$.

Duality. We define

$$\langle u,v\rangle = \int_\Omega u\,v\,dx \tag{53}$$

for any functions such that the right hand side makes sense. For $r\in(1,\infty)$, each element $v\in L^{r'}(\Omega)$, $r'=r/(r-1)$, determines the functional L_v of $(\mathscr{H}_0^{s,r}(\Omega))'$ by the identity $L_v(u)=\langle u,v\rangle$. We introduce the $(-s,r')$-norm of an element $v\in L^{r'}(\Omega)$ to be by definition the norm of the functional L_v, that is

$$\|v\|_{\mathscr{H}^{-s,r'}(\Omega)} = \sup_{\substack{u\in\mathscr{H}_0^{s,r}(\Omega)\\ \|u\|_{\mathscr{H}_0^{s,r}(\Omega)}=1}} |\langle u,v\rangle|. \tag{54}$$

Let $\mathscr{H}^{-s,r'}(\Omega)$ denote the completion of the space $L^{r'}(\Omega)$ with respect to $(-s,r')$-norm. For an integer s, $\mathscr{H}^{-s,r'}(\Omega)$ is topologically and algebraically isomorphic to $(H_0^{s,r}(\Omega))'$. The same conclusion holds true for all $s\in(0,1)$. Moreover, we can identify $\mathscr{H}^{-s,r'}(\Omega)$ with the interpolation space $[L^{r'}(\Omega),H_0^{-1,r'}(\Omega)]_{s,r}$, see e.g., [19]. With this denotations we have the duality principle

$$\|u\|_{\mathscr{H}_0^{s,r}(\Omega)} = \sup_{\substack{v\in C_0^\infty(\Omega)\\ \|v\|_{\mathscr{H}^{-s,r'}(\Omega)}=1}} |\langle u,v\rangle|. \tag{55}$$

With applications to the theory of Navier-Stokes equations in mind, we introduce the smaller dual space defined as follows. We identify the function $v\in L^{r'}(\Omega)$ with the functional $L_v\in(H^{s,r}(\Omega))'$ and denote by $\mathbb{H}^{-s,r'}(\Omega)$ the completion of $L^{r'}(\Omega)$ in the norm

$$\|v\|_{\mathbb{H}^{-s,r'}(\Omega)} := \sup_{\substack{u\in H^{s,r}(\Omega)\\ \|u\|_{H^{s,r}(\Omega)}=1}} |\langle u,v\rangle|. \tag{56}$$

In the sense of this identification the space $C_0^\infty(\Omega)$ is dense in the interpolation space $\mathbb{H}^{-s,r}(\Omega)$. It follows immediately from the definition that

$$\mathbb{H}^{-s,r'}(\Omega)\subset(H^{s,r}(\Omega))'\subset\mathscr{H}^{-s,r'}(\Omega). \tag{57}$$

For an arbitrary bounded domain $\Omega\subset\mathbb{R}^3$ with a Lipschitz boundary, we introduce the Banach spaces

$$X^{s,r} = H^{s,r}(\Omega)\cap H^{1,2}(\Omega), \tag{58}$$

$$Y^{s,r} = H^{s+1,r}(\Omega)\cap H^{2,2}(\Omega), \tag{59}$$

$$Z^{s,r} = \mathscr{H}^{s-1,r}(\Omega)\cap L^2(\Omega) \tag{60}$$

equipped with the norms

$$\|u\|_{X^{s,r}} = \|u\|_{H^{s,r}(\Omega)} + \|u\|_{H^{1,2}(\Omega)}, \tag{61}$$
$$\|u\|_{Y^{s,r}} = \|u\|_{H^{1+s,r}(\Omega)} + \|u\|_{H^{2,2}(\Omega)}, \tag{62}$$
$$\|u\|_{Z^{s,r}} = \|u\|_{\mathscr{H}^{s-1,r}(\Omega)} + \|u\|_{L^2(\Omega)}. \tag{63}$$

It can be easily seen that the embeddings $Y^{s,r} \hookrightarrow X^{s,r} \hookrightarrow Z^{s,r}$ are compact and for $sr > 3$, each of the spaces $X^{s,r}$ and $Y^{s,r}$ is a commutative Banach algebra.

8 Existence and Uniqueness Theory

Denote by E the closed subspace of the Banach space $Y^{s,r}(\Omega)^3 \times X^{s,r}(\Omega)^2$ in the following form

$$E = \{\vartheta = (\mathbf{v}, \pi, \varphi) \colon \mathbf{v} = 0 \text{ on } \partial\Omega, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}}, \quad \Pi\pi = \pi\}, \tag{64}$$

and denote by $\mathscr{B}_\tau \subset E$ the closed ball of radius τ centered at 0. Next, note that for $sr > 3$, elements of the ball $\mathscr{B}_\tau$ satisfy the inequality

$$\|\mathbf{v}\|_{C^1(\Omega)} + \|\pi\|_{C(\Omega)} + \|\varphi\|_{C(\Omega)} \le c_e(r,s,\Omega)\|\vartheta\|_E \le c_e\tau, \tag{65}$$

where the norm in E is defined by

$$\|\vartheta\|_E = \|\mathbf{v}\|_{Y^{s,r}(\Omega)} + \|\pi\|_{X^{s,r}(\Omega)} + \|\varphi\|_{X^{s,r}(\Omega)}. \tag{66}$$

Theorem 2. *Assume that the surface Σ and given vector field $\mathbf{U}$ satisfy emergent field conditions. Furthermore, let σ^*, τ^* be given constants determined in [19], and let positive numbers r, s, σ satisfy the inequalities*

$$1/2 < s \le 1, \quad 1 < r < 3/(2s-1), \quad sr > 3, \ \sigma > \sigma^*. \tag{67}$$

Then there exists $\tau_0 \in (0, \tau^*]$, depending only on $\mathbf{U}, \Omega, r, s, \sigma$, such that for all

$$\tau \in (0, \tau_0], \quad \lambda^{-1}, R \in (0, \tau^2], \quad \|\mathbf{N} - \mathbf{I}\|_{C^2(\Omega)} \le \tau^2, \tag{68}$$

problem (35)-(38), (46), (47), *with* $\mathbf{u}_0$ *given by* (33), *has a unique solution $\vartheta \in B_\tau$. Moreover, the auxiliary function ζ and the constants $\varkappa, m$ admit the estimates*

$$\|\zeta\|_{X^{s,r}} + |\varkappa| \le c, \quad |m| \le c\tau < 1, \tag{69}$$

where the constant c depends only on $\mathbf{U}, \Omega, r, s$ and σ.

9 Material Derivatives of Solutions

Theorem 2 guarantees the existence and uniqueness of solutions to problem (35)-(38), (46), (47) for all $\mathbf{N}$ close to the identity matrix $\mathbf{I}$. The totality of such solutions can be regarded as the mapping from $\mathbf{N}$ to the solution of the Navier-Stokes equations. The natural question is the smoothness properties of this mapping, in particular its differentiability. With application to shape optimization problems in mind, we consider the particular case where the matrices $\mathbf{N}$ depend on the small parameter ε and have representation (19). We assume that C^1 norms of the matrix-valued functions $\mathbf{D}$ and $\mathbf{D}_1(\varepsilon)$ in (19) have a majorant independent of ε. By virtue of Theorem 2, there are the positive constants ε_0 and τ such that for all sufficiently small R, λ^{-1} and $\varepsilon \in [0, \varepsilon_0]$, problem (35)-(38), (46), (47) with $\mathbf{N} = \mathbf{N}(\varepsilon)$ has a unique solution $\vartheta(\varepsilon) = (\mathbf{v}(\varepsilon), \pi(\varepsilon), \varphi(\varepsilon))$, $\zeta(\varepsilon), m(\varepsilon)$, which admits the estimate

$$\|\vartheta(\varepsilon)\|_E + |m(\varepsilon)| \leq c\tau, \quad \|\zeta(\varepsilon)\|_{X^{s,r}} \leq c, \tag{70}$$

where the constant c is independent of ε, and the Banach space E is defined by (64). Denote the solution for $\varepsilon = 0$, $(\vartheta(0), m(0), \zeta(0))$ by (ϑ, m, ζ), and define the finite differences with respect to ε

$$(\mathbf{w}_\varepsilon, \omega_\varepsilon, \psi_\varepsilon) = \varepsilon^{-1}(\vartheta - \vartheta(\varepsilon)),\ \xi_\varepsilon = \varepsilon^{-1}(\zeta - \zeta(\varepsilon)),\ n_\varepsilon = \varepsilon^{-1}(m - m(\varepsilon)). \tag{71}$$

Formal calculations show that the limit $(\mathbf{w}, \omega, \psi, \xi, n) = \lim_{\varepsilon \to 0}(\mathbf{w}_\varepsilon, \omega_\varepsilon, \psi_\varepsilon, \xi_\varepsilon, n_\varepsilon)$ is a solution to linearized equations

$$\begin{aligned}
&\Delta \mathbf{w} - \nabla \omega = R\mathscr{C}_0(\mathbf{w}, \psi) + \mathscr{D}_0(\mathbf{D}) \text{ in } \Omega, \\
&\operatorname{div} \mathbf{w} = b^0_{21}\,\psi - b^0_{22}\,\omega + b^0_{23}\,n + b^0_{30}\,\mathfrak{d} \text{ in } \Omega, \\
&\mathbf{u}\nabla\psi + \sigma\psi = -\mathbf{w}\cdot\nabla\varphi + b^0_{11}\,\psi + b^0_{12}\,\omega + b^0_{13}\,n + b^0_{10}\,\mathfrak{d} \text{ in } \Omega, \\
&-\operatorname{div}(\mathbf{u}\xi) + \sigma\xi = \operatorname{div}(\zeta\mathbf{w}) + \sigma\mathfrak{d} \text{ in } \Omega, \\
&\mathbf{w} = 0 \text{ on } \partial\Omega,\ \psi = 0 \text{ on } \Sigma_{\text{in}},\ \xi = 0 \text{ on } \Sigma_{\text{out}}, \\
&\omega - \Pi\omega = 0,\ n = \varkappa \int_\Omega \Big(b^0_{31}\,\psi + b^0_{32}\,\omega + b^0_{34}\,\xi + b^0_{30}\,\mathfrak{d}\Big)\,dx,
\end{aligned} \tag{72}$$

where $\mathfrak{d} = 1/2\,\mathrm{Tr}\,\mathbf{D}$, the variable coefficients b^0_{ij} and the operators $\mathscr{C}_0$, $\mathscr{D}_0$, are defined by the formulae

$$
\begin{aligned}
b^0_{11} &= \Psi[\vartheta] - \rho H'(\varphi) + m - \frac{2\sigma}{\rho_0}\varphi, \\
b^0_{12} &= \lambda^{-1}\rho, \\
b^0_{13} &= \rho, \\
b^0_{10} &= \rho\Psi[\vartheta] - \frac{\sigma}{\rho_0}\varphi^2 - \sigma\varphi + m\rho, \\
b^0_{21} &= \frac{\sigma}{\rho_0}\psi_0 + H'(\varphi), \\
b^0_{22} &= -\lambda^{-1}, \\
b^0_{23} &= -1, \\
b^0_{20} &= \sigma\varphi\rho_0^{-1} - \Psi[\vartheta] - m, \\
b^0_{31} &= \rho_0^{-1}\zeta\Big(\Psi[\vartheta] - \rho H'(\varphi) - \frac{2\sigma}{\rho_0}\varphi\Big) - H'(\varphi) + m\rho_0^{-1}\zeta, \\
b^0_{32} &= (\lambda\rho_0)^{-1}\rho\zeta b^0_{12} + \lambda^{-1}, \\
b^0_{34} &= \rho_0^{-1}\Psi_1[\vartheta] + m(1+\rho_0^{-1}\varphi) \\
b^0_{30} &= \rho_0^{-1}\zeta(\eth_0 - m\rho) + \Psi[\vartheta] - m(1-\zeta-\rho_0^{-1}\zeta\varphi),
\end{aligned} \tag{73}
$$

$$
\mathscr{C}_0(\psi,\mathbf{w}) = R\psi\mathbf{u}\nabla\mathbf{u} + R\rho\mathbf{w}\nabla\mathbf{u}, + R\rho\mathbf{u}\nabla\mathbf{w}, \tag{74}
$$

$$
\begin{aligned}
\mathscr{D}_0(\mathbf{D}) &= R\mathbf{u}\nabla(\mathbf{D}\mathbf{u}) + R\mathbf{D}^*(\mathbf{u}\nabla\mathbf{u}) \\
&\quad + \operatorname{div}\Big((\mathbf{D}+\mathbf{D}^*)\nabla\mathbf{u} - \frac{1}{2}\mathrm{Tr}\mathbf{D}\nabla\mathbf{u}\Big) - \mathbf{D}\Delta\mathbf{u} - \Delta(\mathbf{D}\mathbf{u}).
\end{aligned} \tag{75}
$$

The justification of the formal procedure meets serious problems, since the smoothness of solutions to problem (35)-(38), (46), (47) is not sufficient for the well-posedness of problem (72) in the standard weak formulation. In order to cope with this difficulty we define *very weak solutions* to problem (72). The construction of such solutions is based on the following lemma [19]. The lemma is given in $\mathbb{R}^d$, for our application $d = 3$.

Lemma 1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with the Lipschitz boundary, let exponents s and r satisfy the inequalities $sr > d$, $1/2 \le s \le 1$ and $\varphi, \varsigma \in H^{s,r}(\Omega) \cap H^{1,2}(\Omega)$, $\mathbf{w} \in \mathscr{H}_0^{1-s,r'}(\Omega) \cap H_0^{1,2}(\Omega)$. Then there is a constant c depending only on s, r and Ω, such that the trilinear form*

$$
\mathfrak{B}(\mathbf{w},\varphi,\varsigma) = -\int_\Omega \varsigma\mathbf{w}\cdot\nabla\varphi\,dx \tag{76}
$$

satisfies the inequality

$$
|\mathfrak{B}(\mathbf{w},\varphi,\varsigma)| \le c\|\mathbf{w}\|_{\mathscr{H}_0^{1-s,r'}(\Omega)}\|\varphi\|_{H^{s,r}(\Omega)}\|\varsigma\|_{H^{s,r}(\Omega)}, \tag{77}
$$

and can be continuously extended to $\mathfrak{B} : \mathscr{H}_0^{1-s,r'}(\Omega)^d \times H^{s,r}(\Omega)^2 \to \mathbb{R}$*. In particular, we have* $\varsigma\nabla\varphi \in \mathscr{H}^{s-1,r}(\Omega)$ *and* $\|\varsigma\nabla\varphi\|_{H^{1-s,r}(\Omega)} \le c\|\varphi\|_{H^{s,r}(\Omega)}\|\varsigma\|_{H^{s,r}(\Omega)}$.

Definition 2. The vector field $\mathbf{w} \in \mathscr{H}_0^{1-s,r'}(\Omega)^3$, functionals $(\omega,\psi,\xi) \in \mathbb{H}^{-s,r'}(\Omega)^3$ and constant n are said to be a weak solution to problem (72), if $\langle\omega,1\rangle = 0$ and the identities

$$\begin{aligned}
&\int_\Omega \mathbf{w}\Big(\mathbf{H} - R\rho\nabla\mathbf{u}\cdot\mathbf{h} + R\rho\nabla\mathbf{h}^*\mathbf{u}\Big)\,dx - \mathfrak{B}(\mathbf{w},\varphi,\varsigma) - \mathfrak{B}(\mathbf{w},\upsilon,\zeta)\\
&+ \big\langle\omega, G - b_{12}^0\varsigma - b_{22}^0 g - \varkappa b_{32}^0\big\rangle + \big\langle\psi, F - b_{11}^0\varsigma - b_{21}^0 g - \varkappa b_{31}^0 - R\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{h}\big\rangle\\
&+ \big\langle\xi, M - \varkappa b_{34}^0\big\rangle + n\big(1 - \big\langle 1, b_{13}^0\varsigma\big\rangle\big)\\
&= \big\langle\mathfrak{d}, b_{10}^0\varsigma + b_{20}^0 g + \varkappa b_{30}^0 + \sigma\upsilon\big\rangle + \big\langle\mathscr{D}_0,\mathbf{h}\big\rangle.
\end{aligned} \tag{78}$$

hold true for all $(\mathbb{H},G,F,M) \in (C^\infty(\Omega))^6$ such that $G = \Pi G$. Here $\mathfrak{d} = 1/2\ \mathrm{Tr}\,\mathbf{D}$, the test functions $\mathbf{h}$, g, ς, υ are defined by the solutions to adjoint problems

$$\Delta\mathbf{h} - \nabla g = \mathbb{H}, \quad \mathrm{div}\,\mathbf{h} = G, \quad \mathscr{L}^*\varsigma = F, \quad \mathscr{L}\upsilon = M \text{ in } \Omega, \tag{79}$$

$$\mathbf{h} = 0 \text{ on } \partial\Omega, \quad \Pi g = g, \varsigma = 0 \text{ on } \Sigma_{\mathrm{out}}, \qquad \upsilon = 0 \text{ on } \Sigma_{\mathrm{in}}. \tag{80}$$

We are now in a position to formulate the third main result of this paper.

Theorem 3. *Under the above assumptions,*

$$\begin{aligned}
&\mathbf{w}_\varepsilon \to \mathbf{w} \text{ weakly in } \mathscr{H}_0^{1-s,r'}(\Omega), \quad n_\varepsilon \to n \text{ in } \mathbb{R},\\
&\psi_\varepsilon \to \psi, \quad \omega_\varepsilon \to \omega, \quad \xi_\varepsilon \to \xi \ (*)\text{-weakly in } \mathbb{H}^{-s,r'}(\Omega) \text{ as } \varepsilon \to 0,
\end{aligned} \tag{81}$$

where the limits, vector field $\mathbf{w}$*, functionals* ψ,ω,ξ*, and the constant* n *are given by the weak solution to problem* (72)*.*

Note that the matrices $\mathbf{N}(\varepsilon)$ defined by equalities (18) meet all requirements of Theorem 3, and in the special case we have in representation (19)

$$\mathbf{D}(x) = \mathrm{div}\,\mathbf{T}(x)\,\mathbf{I} - \mathbf{T}'(x). \tag{82}$$

Therefore, Theorem 3, together with the formulae (15) and (32), imply the existence of the shape derivative for the drag functional at $\varepsilon = 0$. Straightforward calculations lead to the following result.

Theorem 4. *Under the assumptions of Theorem 3, there exists the shape derivative*

$$\frac{d}{d\varepsilon} J_D(S_\varepsilon)\Big|_{\varepsilon=0} = L_e(\mathbf{T}) + L_u(\mathbf{w},\omega,\psi), \tag{83}$$

where the linear forms L_e *and* L_u *are defined by the equalities*

$$
\begin{aligned}
L_e(\mathbf{T}) = &\int_\Omega \operatorname{div}\mathbf{T}(\nabla\mathbf{u}+\nabla\mathbf{u}^* - \operatorname{div}\mathbf{u}\mathbf{I})\mathbf{U}_\infty\,dx \\
&- \int_\Omega \big[\nabla\mathbf{u}+\nabla\mathbf{u}^* - \operatorname{div}\mathbf{u} - q\mathbf{I} - R\rho\mathbf{u}\otimes\mathbf{u}\big]\mathbf{D}\nabla\eta\cdot\mathbf{U}_\infty\,dx \\
&- \int_\Omega \big[\mathbf{D}^*\nabla\mathbf{u}+\nabla\mathbf{u}^*\mathbf{D}+\nabla(\mathbf{D}\mathbf{u})+\nabla(\mathbf{D}\mathbf{u})^*\big]\nabla\eta\cdot\mathbf{U}_\infty\,dx
\end{aligned} \tag{84}
$$

and

$$
\begin{aligned}
L_u(\mathbf{w},\omega,\psi) = &\int_\Omega \mathbf{w}\big[\Delta\eta\mathbf{U}_\infty + R\rho(\mathbf{u}\cdot\nabla\eta)\mathbf{U}_\infty + R\rho(\mathbf{u}\cdot\mathbf{U}_\infty)\nabla\eta\big]\,dx \\
&+ \big\langle\omega,\nabla\eta\cdot\mathbf{U}_\infty\big\rangle + R\big\langle\psi,(\mathbf{u}\cdot\nabla\eta)(\mathbf{u}\cdot\mathbf{U}_\infty)\big\rangle.
\end{aligned} \tag{85}
$$

While L_e depends directly on the vector field $\mathbf{T}$, the linear form L_u depends on the weak solution $(\mathbf{w},\ \psi,\omega)$ to problem (72), thus depends on the *direction* $\mathbf{T}$ in a very implicit manner, which is inconvenient for applications. In order to cope with this difficulty, we define the *adjoint state* $\mathbf{Y} = (\mathbf{h}, g, \varsigma, \upsilon, l)^\top$ given as a solution to the linear equation

$$
\mathfrak{L}\mathbf{Y} - \mathfrak{U}\mathbf{Y} - \mathfrak{V}\mathbf{Y} = \Theta, \tag{86}
$$

supplemented with boundary conditions (80). Here the operators $\mathfrak{L}$, $\mathfrak{U}$, $\mathfrak{V}$ and the vector field Θ are defined by

$$
\mathfrak{L} = \begin{pmatrix} \Delta & -\nabla & 0 & 0 & 0 \\ \operatorname{div} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathscr{L}^* & 0 & 0 \\ 0 & 0 & 0 & \mathscr{L} & 0 \\ 0 & 0 & -\mathbb{B}_{13} & 0 & 1 \end{pmatrix}, \tag{87}
$$

$$
\mathfrak{U} = \begin{pmatrix} 0 & 0 & -\nabla\varphi & -\zeta\nabla & 0 \\ 0 & 0 & \Pi_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{88}
$$

$$
\mathfrak{V} = \begin{pmatrix} R\rho(\nabla\mathbf{u}-\mathbf{u}\nabla) & 0 & 0 & 0 & 0 \\ 0 & -\lambda^{-1}\Pi & 0 & 0 & \varkappa\Pi b^0_{32} \\ R\mathbf{u}\cdot\nabla\mathbf{u} & b^0_{12} & b^0_{11} & 0 & \varkappa b^0_{31} \\ 0 & 0 & 0 & 0 & \varkappa b^0_{34} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{89}
$$

$$
\Theta = (\Delta\eta\mathbf{U}_\infty + R\rho(\nabla\eta\otimes\mathbf{U}_\infty + \mathbf{U}_\infty\otimes\nabla\eta)\mathbf{u}, \Pi(\nabla\eta\cdot\mathbf{U}_\infty), R(\mathbf{u}\nabla\eta)(\mathbf{u}\mathbf{U}_\infty), 0, 0), \tag{90}
$$

$$
\Pi_{2i}(\cdot) = \Pi(b^0_{2i}(\cdot)),\ \mathbb{B}_{13}(\cdot) = \big\langle 1, b^0_{13}(\cdot)\big\rangle. \tag{91}
$$

The following theorem guarantees the existence of the adjoint state and gives the expression of the shape derivative for the drag functional in terms of the vector field **T**.

Theorem 5. *Assume that a given solution* $\vartheta \in \mathscr{B}_\tau$, $(\zeta, m) \in X^{s,r} \times \mathbb{R}$ *to problem* (35)-(38), (46), (47) *meets all requirements of Theorem 2. Then there exists a positive constant* τ_1 *(depending only on* $\mathbf{U}$, Ω *and* r,s*) such that, if* $\tau \in (0, \tau_1]$ *and* $R\lambda^{-1} \leq \tau_1^2$, *then there exists a unique solution* $\mathbf{Y} \in (Y^{s,r})^3 \times (X^{s,r})^3 \times \mathbb{R}$ *to problem* (86), (80). *The form* L_u *has the representation*

$$L_u(\mathbf{w}, \psi, \omega) = \int_\Omega \big[\operatorname{div} \mathbf{T}\big(b_{10}^0 \varsigma + b_{20}^0 g + \sigma \upsilon + \varkappa b_{30}^0 l\big) + \mathscr{D}_0(\operatorname{div} \mathbf{T} - \mathbf{T}')\mathbf{h}\big]\, dx \quad (92)$$

where the coefficients b_{ij}^0 *and the operator* $\mathscr{D}_0$ *are defined by the formulae* (73), (75).

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