New Applications of Variational Analysis to Optimization and Control

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Abstract We discuss new applications of advanced tools of variational analysis and generalized differentiation to a number of important problems in optimization theory, equilibria, optimal control, and feedback control design. The presented results are largely based on the recent work by the author and his collaborators. Among the main topics considered and briefly surveyed in this paper are new calculus rules for generalized differentiation of nonsmooth and set-valued mappings; necessary and sufficient conditions for new notions of linear subextremality and suboptimality in constrained problems; optimality conditions for mathematical problems with equilibrium constraints; necessary optimality conditions for optimistic bilevel programming with smooth and nonsmooth data; existence theorems and optimality conditions for various notions of Pareto-type optimality in problems of multiobjective optimization with vector-valued and set-valued cost mappings; Lipschitzian stability and metric regularity aspects for constrained and variational systems.

1 Introduction

Variational analysis has been recognized as a rapidly growing and fruitful area in mathematics and its applications concerning mainly the study of optimization and equilibrium problems, while also applying perturbation ideas and *variational principles* to a broad class of problems and situations that may be not of a variational nature. It can be viewed as a modern outgrowth of the classical calculus of variations, optimal control theory, and mathematical programming with the focus on *pertur-*

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bation/approximation techniques, sensitivity issues, and applications. We refer the reader to the now classical monograph by Rockafellar and Wets [58] for the key issues of variational analysis in finite-dimensional spaces and to the recent books by Attouch, Buttazzo and Michaelle [1], Borwein and Zhu [7], and Mordukhovich [31, 32] devoted to new aspects of variational analysis in finite-dimensional and infinite-dimensional spaces with numerous applications to different areas of mathematics, engineering, economics, mechanics, computer science, ecology, biology, etc.

One of the most characteristic features of modern variational analysis is the intrinsic presence of *nonsmoothness*, i.e., the necessity to deal with nondifferentiable functions, sets with nonsmooth boundaries, and set-valued mappings. Nonsmoothness naturally enters not only through initial data of optimization-related problems (particularly those with inequality and geometric constraints) but largely via *variational principles* and other optimization, approximation, and perturbation techniques applied to problems with even smooth data. In fact, many fundamental objects frequently appearing in the framework of variational analysis (e.g., the distance function, value functions in optimization and control problems, maximum and minimum functions, solution maps to perturbed constraint and variational systems, etc.) are inevitably of nonsmooth and/or set-valued structures requiring the development of new forms of analysis that involve *generalized differentiation*. Besides the aforementioned books, we refer the reader to the very recent texts by Jeyakumar and Luc [22] and Schirotzek [59], which present new developments on generalized differentiation and their applications to a variety of optimization-related as well as nonvariational problems.

It is important to emphasize that even the simplest and historically earliest problems of *optimal control* are *intrinsically nonsmooth*, in contrast to the classical calculus of variations. This is mainly due to *pointwise constraints* on control functions that often take only discrete values as in typical problems of automatic control, a primary motivation for developing optimal control theory. Optimal control has always been a major source of inspiration as well as a fruitful territory for applications of advanced methods of variational analysis and generalized differentiation; see, e.g., the books by Clarke [9], Mordukhovich [31, 32], and Vinter [60] with the references therein.

In this paper we discuss some new trends and developments in variational analysis and its applications that are based on the 2-volume book by the author [31, 32] and mostly survey more recent and/or brand new results obtained by the author and his collaborators. As mentioned, generalized differentiation lies at the heart of variational analysis and its applications. We systematically develop a *geometric dual-space approach* to generalized differentiation theory revolving around the *extremal principle*, which can be viewed as a local *variational* counterpart of the classical convex separation in nonconvex settings. This principle allows us to deal with *nonconvex* derivative-like constructions for sets (normal cones), set-valued mappings (coderivatives), and extended-real-valued functions (subdifferentials). These constructions are defined directly in dual spaces and, being nonconvex-valued, cannot be generated by any derivative-like constructions in primal spaces (like tangent cones and directional derivatives). Nevertheless, our basic nonconvex constructions enjoy *comprehensive/full calculus*, which happens to be significantly better than those available for their primal and/or convex-valued counterparts. The developed generalized differential calculus based on variational principles provides the *key tools* for various applications.

Observe to this end that *dual objects* (multipliers, adjoint arcs, shadow prices, etc.) have always been at the center of variational theory and applications used, in particular, for formulating the main optimality conditions in the calculus of variations, mathematical programming, optimal control, and economic modeling. The usage of variations of optimal solutions in primal spaces can be considered just as a convenient tool for deriving necessary optimality conditions. There are no essential restrictions in such a "primal" approach in smooth and convex frameworks, since primal and dual derivative-like constructions are equivalent for these classical settings. It is not the case any more in the framework of modern variational analysis, where even *nonconvex primal space* local approximations (e.g., tangent cones) inevitably yield, *under duality, convex sets* of normals and subgradients. This convexity of dual objects leads to significant restrictions for the theory and applications. Moreover, there are many situations particularly identified in [31, 32], where primal space approximations simply cannot be used for variational analysis, while the employment of dual space constructions provides comprehensive treatments and results.

The main attention of this paper is paid to the description of certain basic constructions of generalized differentiation in variational analysis and their applications to important and also new classes of problems in *constrained optimization* and *optimal control* that happen to be intrinsically *nonsmooth*, even in the case of smooth initial data. In Sect. 2 we define these *dual-space generalized differential constructions* and discuss new *calculus results* for them. Sect. 3 is devoted to recent applications of the generalized differential calculus to studying the notion of *linear suboptimality* in constrained optimization, where the usage of these generalized differential constructions allows us to *fully characterize* linearly suboptimal solutions, in the sense of deriving verifiable *necessary and sufficient* conditions for them.

In Sect. 4 we discuss new results for a broad class of optimization problem known as *mathematical programs with equilibrium constraints* (MPECs) significant in optimization theory and its applications. Besides characterizations of the aforementioned notion of linear suboptimality for MPECs, we present new necessary optimality conditions for the conventional notion of optimal solutions to MPECs whose equilibrium constraints are governed by parameterized *quasivariational inequalities* that are challenging in the MPEC theory and highly important for applications.

Sect. 5 is devoted to new results on the so-called *bilevel programming*, which is a remarkable class of *hierarchical optimization* problems somehow related to MPECs while generally independent. Concentrating on the *optimistic version* of bilevel programs and using our basic tools of generalized differentiation, we present advanced necessary optimality conditions in finite-dimensional bilevel programming that are new even for problems with *smooth* data on both lower level and upper levels.

Sect. 6 concerns various problems of *multiobjective optimization* and *equilibria*, which are among the most challenging theoretically and the most important for numerous applications (to economics, mechanics, and other areas). We pay the main attention to new existence theorems and necessary optimality conditions for Pareto-type solutions to constrained multiobjective problems with vector-valued and set-valued objectives. Our approach is based on developing and implementing advanced variational principles for multifunctions with values in partially ordered spaces.

In Sect. 7 we consider several important issues revolving around *Lipschitzian stability* and *metric regularity* properties for set-valued mappings and their applications to structural systems arising in numerous aspects of variational analysis, optimization, and control. Our approach is based on the *dual coderivative criteria* for such properties established earlier by the author; they can be applied to a variety of structural systems due to well-developed *coderivative calculus* in finite-dimensional and infinite-dimensional spaces. In this way, along with deriving positive results in this direction, we come up to a rather surprising conclusion that major classes of variational/optimality systems, which are the most interesting from the both viewpoints of the theory and applications, *do not exhibit metric regularity.*

Sect. 8 presents new results on *optimal control* dealing mainly with evolution systems governed by constrained *difference*, *differential*, and *delay-differential inclusions* in infinite-dimensional spaces. We develop the method of *discrete approximations* for continuous-time evolution systems and investigate both *qualitative* and *quantitative* aspects of this approach. Our results include *stability/convergence* of discrete approximations, deriving necessary optimality conditions for discrete-time systems and then for the original continuous-time control problems by passing to the limit from discrete approximations and employing advanced tools of variational analysis and generalized differentiation.

The concluding Sect. 9 is devoted to problems of *feedback control design* of *constrained parabolic systems* in *uncertainty conditions*. Control problems of these type are undoubtedly among the most important for various (in particular, engineering and ecological) applications; at the same time they are among the most challenging in control theory. Especially serious difficulties arise in studying and solving such problems in the presence of *hard/pointwise constraints* on control and state variables, which is the case considered in the concluding section motivated by some practical applications to environmental systems. The approach discussed in Sect. 9 and the results presented therein are based on certain specific features of the parabolic dynamics related to *monotonicity* and *turnpike* behavior on the *infinite horizon*, as well as on approximation techniques typical in variational analysis. In this way we justify implementable *suboptimal* structures of *feedback control regulators* acting through boundary conditions and compute their optimal parametric ensuring the *best* behavior of the systems under *worst* perturbations and *robust stability* of the closed-loop systems for arbitrary perturbations from the feasible area.

Throughout the paper we use the standard notation of variational analysis; see, e.g., [31, 58]. Recall that $\mathbb{B}$ stands for the closed unit ball of the space in question and that $\mathbb{N} := \{1,2,\ldots\}$. Given a set-valued mapping $F\colon X \rightrightarrows X^*$ between a Banach space X and its topological dual X^*, the symbol

$$\mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \Big|\, \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in \mathbb{N}\Big\} \tag{1}$$

signifies the *sequential Painlevé-Kuratowski upper/outer limit* of F at $\bar{x}$ in the norm topology of X and weak* topology of X^*.

2 Generalized Differentiation

In this section we define, for the reader's convenience, some basic constructions and properties from variational analysis and generalized differentiation needed in what follows. All these are taken from the book by Mordukhovich [31], where the reader can find more details, discussions, and references. The reader may also consult with the books by Borwein and Zhu [7], Rockafellar and Wets [58], and Schirotzek [59] for related and additional material.

Most results presented in this paper are obtained in the framework of Asplund spaces; so our *standing assumption* is that all the spaces under consideration are *Asplund* unless otherwise stated. One of the equivalent descriptions of an Asplund space is that it is a Banach space for which every separable subspace has a separable dual. It is well known that any reflexive Banach space is Asplund as well as any space with a separable dual; see [31, Sect. 2.2] for more discussions and references. The generalized differential constructions and properties presented below generally rely on the Asplund structure; see [7, 20, 31] for the corresponding modifications in other (including arbitrary) Banach space settings.

Given a nonempty set $\Omega \subset X$, define the *Fréchet normal cone* to Ω at $\bar{x} \in \Omega$ by

$$\hat{N}(\bar{x};\Omega) := \left\{x^* \in X^* \Big|\, \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \le 0\right\}, \tag{2}$$

where the symbol $x \xrightarrow{\Omega} \bar{x}$ signifies that $x \to \bar{x}$ with $x \in \Omega$. Construction (2) looks as an adaptation of the idea of Fréchet derivative to the case of sets; that's where the name comes from. However, this construction does not have a number of natural properties expected for an appropriate notion of normals. In particular, we may have $\hat{N}(\bar{x};\Omega) = \{0\}$ for boundary points of Ω even in simple finite-dimensional nonconvex settings; furthermore, inevitable required calculus rules often fail for (2). The situation is dramatically improved while applying the regularization procedure

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{x \xrightarrow{\Omega} \bar{x}} \hat{N}(x;\Omega) \tag{3}$$

via the sequential outer limit (1) in the norm topology of X and the weak* topology of X^*. The construction (3) is known as the (*basic, limiting, Mordukhovich*) *normal cone* to Ω at $\bar{x} \in \Omega$; it was introduced in [27] in an equivalent form in finite dimensions. Both constructions (2) and (3) reduce to the classical normal cone of convex analysis for convex sets Ω. In contrast to (2), the basic normal cone (3) is often *nonconvex* while satisfying the required properties and *calculus rules* in the Asplund space setting, together with the corresponding coderivative constructions for set-valued mappings and subdifferential constructions for extended-real-valued functions generated by it; see below. All this calculus and the required properties are mainly due to the *extremal/variational principles* of variational analysis; see [31] for more discussions.

Given a set-valued mapping/multifunction $F\colon X \rightrightarrows Y$ with the *graph*

$$\operatorname{gph} F := \big\{(x,y) \in X \times Y \big|\, y \in F(x)\big\}, \tag{4}$$

and following the pattern introduced in [28], define the *coderivative* constructions for F used in this paper. The *Fréchet coderivative* of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is given by

$$\hat{D}^* F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\, (x^*,-y^*) \in \hat{N}\big((\bar{x},\bar{y});\operatorname{gph} F\big)\big\},\ \ y^* \in Y^*, \tag{5}$$

and the *normal coderivative* of F at the reference point is given by

$$D^*_N F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\, (x^*,-y^*) \in N\big((\bar{x},\bar{y});\operatorname{gph} F\big)\big\},\ \ y^* \in Y^*. \tag{6}$$

We also need the following modification of the normal coderivative (5) called the *mixed coderivative* of F at $(\bar{x},\bar{y})$ and defined by

$$\begin{aligned} D^*_M F(\bar{x},\bar{y})(y^*) := \Big\{x^* \in X^* \Big|\, &\exists\, (x_k,y_k) \stackrel{\operatorname{gph} F}{\to} (\bar{x},\bar{y}),\ x^*_k \stackrel{w^*}{\longrightarrow} x^*,\ y^*_k \stackrel{\|\cdot\|}{\longrightarrow} y^* \\ &\text{with}\ \ (x^*_k,-y^*_k) \in \hat{N}\big((x_k,y_k);\operatorname{gph} F\big),\ k \in \mathbb{N}\Big\}, \end{aligned} \tag{7}$$

where $\stackrel{\|\cdot\|}{\to}$ stands for the norm convergence in the dual space; we usually omit the symbol $\|\cdot\|$ indicating the norm convergence simply by "$\to$" and also skip $\bar{y} = f(\bar{x})$ in the coderivative notation if $F = f\colon X \to Y$ is a single-valued mapping. Clearly $D^*_M F(\bar{x},\bar{y}) = D^*_N F(\bar{x},\bar{y})$ if $\dim Y < \infty$, and then we use the same notation D^* for both coderivatives. The above equality also holds in various infinite-dimensional settings, while not in general; see [31, Subsect. 1.2.1 and 4.2.1]. If $F = f$ is single-valued and *smooth* around $\bar{x}$ (or merely *strictly differentiable* at this point), then we have the representations

$$\hat{D}^* f(\bar{x})(y^*) = D^*_M f(\bar{x})(y^*) = D^*_N f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^* y^*\big\}, \quad y^* \in Y^*, \tag{8}$$

which show that the *co*derivative notion is a natural extension of the *adjoint derivative* operator to nonsmooth and set-valued mappings.

Given an extended-real-valued function $\varphi\colon X \to \overline{\mathbb{R}} := (-\infty,\infty]$, consider the associated *epigraphical multifunction* $\mathscr{E}_\varphi\colon X \rightrightarrows \mathbb{R}$ and define the *Fréchet/regular subdifferential* of φ at $\bar{x} \in \operatorname{dom}\varphi$ in the two equivalent (geometric and analytic) ways

$$\hat{\partial}\varphi(\bar{x}) := \hat{D}^*\mathscr{E}_\varphi\big(\bar{x},\varphi(\bar{x})\big)(1) = \Big\{x^* \in X^* \Big|\ \liminf_{x\to\bar{x}} \frac{\varphi(x)-\varphi(\bar{x})-\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \ge 0\Big\}. \tag{9}$$

The *basic/limiting/Mordukhovich subdifferential* of φ at $\bar{x}$ is defined by

$$\partial\varphi(\bar{x}) = D^*\mathscr{E}_\varphi\big(\bar{x},\varphi(\bar{x})\big)(1) = \mathop{\mathrm{Lim\,sup}}_{x \xrightarrow{\varphi} \bar{x}} \hat{\partial}\varphi(x), \tag{10}$$

where the symbol $x \xrightarrow{\varphi} \bar{x}$ stands for $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$. Note that the Fréchet subdifferential agrees with the Crandall-Lions subdifferential in the sense of *viscosity solutions* to partial differential equations independently introduced in [10], while the limiting construction (10) reduces to that introduced in [27] motivated by applications to optimal control. The *convexification* of (10) for locally Lipschitzian functions agrees with the generalized gradient introduced by Clarke via different relationships; see [9]. For non-Lipschitzian functions φ it makes sense to consider the *singular* counterpart of φ given by

$$\partial^\infty\varphi(\bar{x}) = D^*\mathscr{E}_\varphi\big(\bar{x},\varphi(\bar{x})\big)(0) = \mathop{\mathrm{Lim\,sup}}_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \lambda\downarrow 0}} \lambda\hat{\partial}\varphi(x), \tag{11}$$

which reduces to $\{0\}$ if φ is locally Lipschitzian around $\bar{x}$.

Among the main advantages of the robust limiting constructions (3), (6), (7), (10), and (11), we particularly mention *full pointwise calculi* available for them, the possibility to *characterize* in their terms *Lipschitzian, metric regularity*, and *openness* properties of set-valued and single-valued mappings that play a fundamental role in nonlinear analysis and its applications, and to derive in their terms refined conditions for *optimality* and *sensitivity* in various problems of optimization, equilibria, control, etc. Besides variational principles, extended *calculus is the key* for major theoretical advances and applications.

Referring the reader to [31] for a variety of calculus rules for the basic normals, subgradients, and coderivatives under consideration, let us mention several recent ones (in addition to [31]) motivated by the required applications presented in the corresponding papers.

In [18], we develop certain calculus rules for the so-called *reversed mixed coderivative* of $F\colon X \rightrightarrows Y$ at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ defined by

$$\tilde{D}^*_M F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ -y^* \in D^*_M F^{-1}(\bar{y},\bar{x})(-x^*)\big\}, \tag{12}$$

which is different from both coderivative constructions (6) and (7) in infinite dimensions while playing a crucial role in characterizing *metric regularity*. In contrast to (6) and (7), the reversed construction (12) does *not* generally enjoy satisfactory calculus rules, since taking the inverse in (12) dramatically complicates some major operations (e.g., sums) for single-valued and set-valued mappings. The calculus rules derived in [18] for the reversed coderivative (12) mainly address a special class of set-valued mappings known as solution maps to *generalized equations* (in the sense of Robinson [57]):

$$S(x) = \{y \in Y \,|\, 0 \in f(x,y) + Q(y)\} \text{ with } f\colon X \times Y \to Z \text{ and } Q\colon Y \rightrightarrows Z, \tag{13}$$

which are highly important in many aspects of variational analysis and optimization; see, e.g., [17, 31, 56, 58] and the references therein. The calculus results obtained in [18] and related developments allow us to make a principal conclusion on the *failure of metric regularity* for major classes of parametric *variational systems*; see Sect. 7 below.

Another important setting that requires new coderivative calculus rules is described by set-valued mappings in the form

$$Q(x,y) = N\big(y; \Lambda(x,y)\big) \text{ with } \Lambda\colon X \times Y \rightrightarrows Y, \tag{14}$$

which corresponds to the so-called *quasivariational inequalities* in the generalized equation framework (13) with $Q = Q(x,y)$ of type (14). Advanced results in this direction are obtained in [50] in finite-dimensional spaces and are applied there to *sensitivity analysis* of quasivariational inequalities and *necessary optimality conditions* for the corresponding MPECs; see Sect. 4 and 7 for more details.

Let us also mention new *intersection rules* for coderivatives obtained in [46] in general infinite-dimensional settings and applied therein to sensitivity analysis of extended parametric models of type (13) arising in various applications, particularly to *bilevel programs*; see Sect. 5 for more discussions.

Several new calculus rules for the (basic and singular) *limiting subdifferentials* (10) and (11) of the important classes of *marginal/value functions* are derived in [49] with applications to sensitivity analysis and optimality conditions in problems of mathematical programming in finite-dimensional and infinite-dimensional spaces. In [48], rather surprising *exact* (versus "fuzzy") calculus rules are obtained for the *Fréchet subdifferential* (9) of various compositions and marginal functions with applications to some classes of optimization problems; see Sect. 3. Among them the following striking *difference rule*:

$$\hat\partial(\varphi_1 - \varphi_2)(\bar x) \subset \bigcap_{x^* \in \hat\partial\varphi_2(\bar x)} \Big[\hat\partial\varphi_1(\bar x) - x^*\Big] \subset \hat\partial\varphi_1(\bar x) - \hat\partial\varphi_2(\bar x) \tag{15}$$

is derived in general Banach spaces provided that $\hat\partial\varphi_2(\bar x) \neq \emptyset$. Counterparts of such exact calculus results for the so-called *proximal subgradients* can be found in [45].

3 Constrained Optimization

It has been well recognized that, except convex programming and related problems with a convex structure, *necessary* conditions are usually *not sufficient* for conventional notions of optimality. Observe also that major necessary optimality conditions in all the branches of the classical and modern optimization theory (e.g., Lagrange multipliers and Karush-Kuhn-Tucker conditions in nonlinear programming, the Euler-Lagrange equation in the calculus of variations, the Pontryagin maximum principle in optimal control, etc.) are expressed in *dual* forms involving *adjoint* variables. At the same time, the very *notions of optimality*, in both scalar and vector frameworks, are formulated of course in *primal* terms.

A challenging question is to find certain modified notions of local optimality so that *first-order* necessary conditions known for the previously recognized notions become *necessary and sufficient* in the new framework. Such a study has been initiated by Kruger (see [25] and the references therein), where the corresponding notions are called "weak stationarity". It seems that the main difference between the conventional notions and those of the type [25] is that the latter relate to a certain *suboptimality* not at the point in question but in a *neighborhood* of it, and that they involve a *linear rate* similar to that in *Lipschitz continuity* (in contrast merely to continuity) as well as in modern concepts of *metric regularity* and *linear openness*, which distinguishes them from the classical regularity and openness notions of nonlinear analysis. On this basis we suggested in [32] to use the names of *linear subextremality* for set systems and of *linear suboptimality* for the corresponding notions in optimization problems.

As has been fully recognized just in the framework of modern variational analysis (even regarding the classical settings), the *linear rate nature* of the fundamental properties involving Lipschitz continuity, metric regularity, and openness for single-valued and set-valued mappings is the *key issue* allowing us to derive *complete characterizations* of these properties via appropriate tools of generalized differentiation; see the books [31, 58] and their references. Precisely the same linear rate essence of the (sub)extremality and (sub)optimality concepts considered below is the driving force ensuring the possibility to justify the validity of known necessary extremality and optimality conditions for the conventional notions as *necessary and sufficient* conditions for the new notions under consideration.

In contrast to [25], where dual criteria for "weak stationarity" are obtained in "fuzzy" forms involving Fréchet-like constructions at points *nearby* the reference ones, in [32, Chapter 5] and in the more recent developments [36, 38] we pay the main attention to *pointwise* conditions expressed via the basic *robust* generalized differential constructions discussed in Sect. 2, which are defined *exactly at* the points in question. Besides the latter being more convenient for applications, we can significantly gain from such pointwise characterizations due to the well-developed *full calculus* enjoyed by the robust constructions, which particularly allows us to cover problems with various *constrained* structures important for both the optimization theory and its applications.

A major role in our approach to variational analysis and optimization systematized and developed in [31, 32] is played by the so-called *extremal principle*; see [31, Chapter 2] with the references and comprehensive discussions therein. Recall that a point $\bar{x} \in \Omega_1 \cap \Omega_2 \subset X$ is *locally extremal* for the set system $\{\Omega_1, \Omega_2\}$ if there exists a neighborhood U of $\bar{x}$ such that for any $\varepsilon > 0$ there is $a \in \varepsilon\mathbb{B}$ with

$$(\Omega_1 + a) \cap \Omega_2 \cap U = \emptyset. \tag{16}$$

Loosely speaking, the local extremality of sets at a common point means that they can be locally "pushed apart" by a small perturbation/translation of one of them. It has been well recognized that set extremality encompasses various notions of optimal solutions to problems of scalar and vector/multiobjective optimization, equilibria, etc.

It is easy to observe that $\bar{x} \in \Omega_1 \cap \Omega_2$ is locally extremal for $\{\Omega_1, \Omega_2\}$ *if and only if*

$$\vartheta\big(\Omega_1 \cap B_r(\bar{x}), \Omega_2 \cap B_r(\bar{x})\big) = 0 \text{ with some } r > 0, \tag{17}$$

where $B_r(\bar{x}) := \bar{x} + r\mathbb{B}$, and where the *measure of overlapping* $\vartheta(\Omega_1, \Omega_2)$ for the sets Ω_1, Ω_2 is defined by

$$\vartheta(\Omega_1, \Omega_2) := \sup\big\{\nu \geq 0 \big|\, \nu\mathbb{B} \subset \Omega_1 - \Omega_2\big\}. \tag{18}$$

Following [25] and the terminology in [32, Sect. 5.4], we say that the set system $\{\Omega_1, \Omega_2\}$ is *linearly subextremal* around $\bar{x} \in \Omega_1 \cap \Omega_2$ if

$$\vartheta_{\rm lin}(\Omega_1, \Omega_2, \bar{x}) := \liminf_{\substack{x_i \xrightarrow{\Omega_i} \bar{x} \\ r \downarrow 0}} \frac{\vartheta\big([\Omega_1 - x_1] \cap r\mathbb{B}, [\Omega_2 - x_2] \cap r\mathbb{B}\big)}{r} = 0 \tag{19}$$

with $i = 1, 2$ under the "lim inf" sign in (19); see [25, 32, 36] for more discussions.

To formulate the following results about the extremal principle also for the subsequent use in the paper, recall that a set $\Omega \subset X$ is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequences $x_k \xrightarrow{\Omega} \bar{x}$ and $x_k^* \xrightarrow{w^*} 0$ we have

$$\|x_k^*\| \to 0 \;\text{ provided that }\; x_k^* \in \hat{N}(x_k; \Omega) \;\text{ as }\; k \to \infty. \tag{20}$$

In finite dimensions, every subset is obviously SNC. For arbitrary Banach space, Ω is SNC at $\bar{x}$ if it is "compactly epi-Lipschitzian" in the sense of Borwein and Strójwas; see [31, Subsect. 1.1.4] for this and other sufficient conditions. If Ω is *convex* in infinite dimensions, then its SNC property is closely related to Ω being of *finite codimension*.

The *extremal principle* from [31, Theorem 2.20] says that for any local extremal point $\bar{x} \in \Omega_1 \cap \Omega_2$ of the system $\{\Omega_1, \Omega_2\}$ of closed subsets of an Asplund space X there is $x^* \in X^*$ satisfying the relationship

$$0 \neq x^* \in N(\bar{x}; \Omega_1) \cap \big(-N(\bar{x}; \Omega_2)\big) \tag{21}$$

provided that either Ω_1 or Ω_2 is SNC at $\bar{x}$. This result can be treated as a *variational counterpart* of the classical convex separation theorem in *nonconvex* settings. In fact, its role in variational analysis is similar to that of convex separation in convex analysis and its "convexified" versions; see [31, 32] for more details and discussions.

An appropriate “necessary and sufficient” modification of the extremal principle for *linear subextremality* reads as follows; cf. [32, Theorem 5.89] and [36, Theorem 1].

Theorem 1. (Necessary and sufficient conditions for linear subextremality via the extremal principle). *Let Ω_1 and Ω_2 be subsets of an Asplund space X that are locally closed around $\bar{x} \in \Omega_1 \cap \Omega_2$. If the system $\{\Omega_1, \Omega_2\}$ is linearly suboptimal around $\bar{x}$, then there is $x^* \in X^*$ satisfying the extremal principle* (21). *Furthermore, the extremal principle* (21) *is necessary and sufficient for the linear suboptimality of $\{\Omega_1, \Omega_2\}$ around $\bar{x}$ if* $\dim X < \infty$.

Based on this theorem and on well-developed *robust calculus* rules for our *limiting* generalized differential constructions, we derive in [32, Sect. 5.4], [36, 38] a number of *necessary* as well as *necessary and sufficient* conditions for the notions of *linear suboptimality* generated by the set subextremality (19) for various optimization and equilibrium problems involving constraints of geometric, operator, functional, and equilibrium types. It should be emphasized that to derive in this way necessary and sufficient conditions for constraint problems, we need to use generalized differential results ensuring *equalities* in the corresponding calculus rules. Such results are largely available in [31] and are employed in [32, 36, 38].

Among other recent applications to optimization, let us mention new necessary optimality conditions for *sharp minimizers* and also to *DC* (*difference of convex*) *programs* derived in [45, 48, 49] on the basis of the subdifferential calculus rules developed therein in both finite-dimensional and infinite-dimensional settings.

A series of new results on necessary conditions for nonsmooth *infinite-dimensional* optimization problems are established in [35] based on advanced methods of variational analysis, on extended calculus rules of *generalized differentiation* as well as on efficient *calculus/preservation* rules for the *sequential normal compactness property* (20) and its *partial* counterparts. These results include several new versions of the *Lagrange principle* for nonsmooth optimization problems with functional and geometric constraints and also refined necessary conditions for problems with *operator constrains* given by nonsmooth *Fredholm-type* mappings with values in infinite dimensions. The latter result is applied to constrained *optimal control* problems governed by *discrete-time inclusions*; see Sect. 8 for more details.

4 Mathematical Programs with Equilibrium Constraints

The modern terminology of *mathematical programs with equilibrium constraints* (MPECs) generally concerns optimization problems given in the following form:

$$\text{minimize } \varphi_0(x,y) \text{ subject to } y \in S(x), \ (x,y) \in \Omega, \tag{22}$$

which contain, among other constraints, the so-called *equilibrium constraints* defined by solution maps to the parameterized *generalized equations/variational conditions*

$$S(x) := \big\{ y \in Y \big|\, 0 \in f(x,y) + Q(x,y) \big\} \tag{23}$$

that are described by single-valued *base* mappings $f\colon X \times Y \to Z$ and set-valued *field* mappings $Q\colon X \times Y \rightrightarrows Z$; see, e.g., [17, 31, 56] for more discussions. Variational systems of type (23) are introduced in the seminal work by Robinson [57] in the setting when $Q(y) = N(y;\Lambda)$ is the *normal cone mapping* to a *convex* set Λ, in which case the generalized equation (23) reduces to the parametric *variational inequality*:

$$\text{find } \; y \in \Lambda \; \text{ such that } \; \big\langle f(x,y), v - y \big\rangle \ge 0 \; \text{ for all } \; v \in \Lambda. \tag{24}$$

The classical parametric *complementarity system* corresponds to (24) when Λ is the nonnegative orthant in $\mathbb{R}^n$. It is well known that the latter model covers sets of *optimal solutions* with the associated *Lagrange multipliers* and sets of *Karush-Kuhn-Tucker* (KKT) vectors satisfying first-order necessary optimality conditions in parametric problems of *nonlinear programming* with *smooth* data. General models with parameter-dependent field mappings $Q = Q(x,y)$ in (23) have been also, but to much lesser extent, considered in the literature. They are related, in particular, to the *quasivariational inequalities*

$$\text{find } \; y \in \Lambda \; \text{ such that } \; \big\langle f(x,y), v - y \big\rangle \ge 0 \; \text{ for all } \; v \in \Lambda(x,y) \tag{25}$$

in the extended framework of (24); see [50] for more discussions and references. Note that in *infinite-dimensional* spaces models of these types are closely associated with variational problems arising in *partial differential equations*.

Variational systems most important for optimization/equilibrium theory and applications mainly relate to generalized equations (23) with *subdifferential fields* when Q is given by a *subdifferential/normal cone operator* $\partial\varphi$ generated by an extended-real-valued lower semicontinuous (l.s.c.) function φ, which is often labeled as *potential*. As mentioned above, this is the case of the classical variational inequalities (24) and complementarity problems generated by convex *indicator* functions $\varphi(\cdot) = \delta(\cdot;\Lambda)$ as well as of their quasivariational counterparts in (25). Formalism (23) with $Q = \partial\varphi$ encompasses also other types of variational and extended variational inequalities generated by *nonconvex* potentials, e.g., the so-called *hemivariational inequalities* with Lipschitzian potentials.

In this vein, two remarkable classes of equilibrium constraints are of particular interest for optimization/equilibrium theory and applications. The first one is given in the form

$$S(x) := \big\{ y \in Y \big|\, 0 \in f(x,y) + \partial(\psi \circ g)(x,y) \big\}, \tag{26}$$

where $g\colon X \times Y \to W$ and $f\colon X \times Y \to X^* \times Y^*$ are single-valued mappings between Banach spaces, and where $\partial\varphi\colon X \times Y \rightrightarrows X^* \times Y^*$ is the basic subdifferential mapping (10) generated by the *composite potential* $\varphi = \psi \circ g$ with $\psi\colon W \to \bar{\mathbb{R}}$. The aforementioned variational systems are special cases of the composite formalism (26).

The second class of remarkable equilibrium constraints is described by the generalized equations with *composite subdifferential fields*

$$S(x) := \big\{ y \in Y \big|\, 0 \in f(x,y) + (\partial \psi \circ g)(x,y) \big\}, \tag{27}$$

where $g\colon X \times Y \to W$, $\psi\colon W \to \bar{\mathbb{R}}$, and $f\colon X \times Y \to W^*$. Formalism (27) encompasses, in particular, perturbed *implicit complementarity problems* of the type: find $y \in Y$ satisfying

$$f(x,y) \geq 0, \quad y - g(x,y) \geq 0, \quad \langle f(x,y), y - g(x,y) \rangle = 0, \tag{28}$$

where the inequalities are understood in the sense of some order on Y.

It occurs nevertheless that generalized equation and variational inequality models of the types discussed above with *single-valued base* mappings $f(x,y)$ do not cover a number of variational systems important in optimization theory and applications. Consider, e.g., the *parametric optimization* problem

$$\text{minimize } \; \phi(x,y) + \vartheta(x,y) \; \text{ over } \; y \in Y \tag{29}$$

described by a *cost* function ϕ and a *constraint* function ϑ that generally take their values in the extended real line $\bar{\mathbb{R}}$. The *stationary point multifunction* associated with (29) is

$$S(x) := \big\{ y \in Y \big|\, 0 \in \partial_y \phi(x,y) + \partial_y \vartheta(x,y) \big\} \tag{30}$$

via collections of partial subgradients of the cost and constraint functions with respect to the decision variable. If the cost function ϕ in (29) is *smooth*, then $\partial_y \phi(x,y) = \{\nabla_y \phi(x,y)\}$ and thus (30) can be written as the solution map to a generalized equation of type (23) with the base $f(x,y) = \nabla_y \phi(x,y)$ and the field mapping $Q(x,y) = \partial_y \vartheta(x,y)$. However, in the case of *nonsmooth optimization* in (29) corresponding, e.g., to *nonsmooth bilevel programs* (see Sect. 5), the stationary point multifunction (30) cannot be written as the standard generalized equation (23) while requiring the *extended formalism*

$$0 \in F(x,y) + Q(x,y), \tag{31}$$

where both the base mapping F and the field mapping Q are *set-valued*.

Another interesting and important class of variational systems that can be written in the extended generalized equation form (31) but not in the conventional one (23) is described by the so-called *set-valued/generalized variational inequalities*:

$$\text{find } \; y \in \Omega \; \text{ such that } \; y^* \in F(x,y) \; \text{ with } \; \big\langle y^*, v - y \big\rangle \geq 0 \; \text{ for } \; v \in \Lambda, \tag{32}$$

which provide a set-valued extension of (24); see, e.g., the handbook [61] for the theory and applications of (32) and related models.

In the recent papers [2, 4, 37, 38, 46, 50, 51] we derive *necessary optimality conditions* for various MPECs (22) as well as for related multiobjective optimization and equilibrium problems with equilibrium constraints governed by generalized equations/variational conditions (23)–(27), (30)–(32) and their specifications. A major role in these conditions is played by the *Fredholm constraint qualification*, which reads, in the particular case of the generalized equation in (22) with a smooth base, as that the *adjoint generalized equation*

$$0 \in \nabla f(\bar{x},\bar{y})^* z^* + D^*Q(\bar{x},\bar{y},\bar{z})(z^*) \ \text{ with } \ \bar{z} := -f(\bar{x},\bar{y}) \tag{33}$$

has only the *trivial solution* $z^* = 0$. Furthermore, in [38] we derive *necessary and sufficient* conditions for *linear suboptimality* in some of such problems.

Following the pattern developed in [32, Sect. 5.2], the results obtained in the aforementioned papers are generally expressed via *coderivatives* of the base and/or field mappings, while for *subdifferential systems* of types (26) and (27) we employ the *second-order subdifferentials* of extended-real-valued functions defined by the scheme

$$\partial^2\varphi(\bar{x},\bar{y}) := (D^*\partial\varphi)(\bar{x},\bar{y}) \ \text{ for } \ \bar{y} \in \partial\varphi(\bar{x}) \tag{34}$$

via the corresponding coderivatives of the first-order subdifferential mappings; see [31] and the references therein for more details, calculus rules, explicit computations, and a number of applications of the second-order subdifferential constructions.

5 Bilevel Programming

Bilevel programming deals with a broad class of problems in *hierarchical optimization* that consist of minimizing *upper-level* objective functions subject to upper-level constraints given by set-valued mappings whose values are sets of *optimal solutions* to some *lower-level* problems of parametric optimization. There are several frameworks of bilevel programs and a number of approaches to their study and applications; see the book [11] and the extended introduction to [12] for more discussions and references. The so-called *optimistic version* in bilevel programming reads as follows:

$$\text{minimize } \ \varphi_0(x) \text{ subject to } x \in \Omega \text{ with } \ \varphi_0(x) := \inf\big\{\varphi(x,y)\big|\ y \in \Psi(x)\big\}, \tag{35}$$

where the sets $\Psi(x)$ of *feasible solutions* to the upper-level problem in (35) consist of *optimal solutions* to the parametric lower-level optimization problem

$$\Psi(x) := \text{argmin}\big\{\psi(x,y)\big|\ f_i(x,y) \le 0,\ i = 1,\ldots,m\big\}, \tag{36}$$

which may also contain constraints of other types (e.g., given by equalities).

Note that problems of this type are *intrinsically nonsmooth*, even for smooth initial data, and can be treated by using appropriate tools of modern variational analysis and generalized differentiation. In [12], we develop the so-called *value function approach* to bilevel programs in (35) and (36) that reduces them to the single-level framework of nondifferentiable programming formulated via (nonsmooth) optimal value functions of parametric lower-level problems in the original model.

It is important to observe that standard *constraint qualifications* in mathematical programming (e.g., the classical Mangasarian-Fromovitz one and the like) are *violated* for single-level programs obtained in this way. An appropriate qualification condition for bilevel programs related to a certain exact penalization was introduced in [62] under the name of "partial calmness". Using the latter constraint qualification and advanced formulas for computing and estimating *limiting subgradients* of *value/marginal functions* in parametric optimization obtained in [31, 49], we derive new necessary optimality conditions for bilevel programs reflecting significant phenomena that have never been observed earlier. In particular, the necessary optimality conditions for bilevel programs established in [12] do *not depend* on the *partial derivatives* with respect to *parameters* of smooth objective functions in parametric lower-level problems. Efficient implementations of this approach are developed in [12] for bilevel programs with differentiable, convex, linear, and locally Lipschitzian functions describing the initial data of lower-level and upper-level problems.

The results obtained in [12] have been recently improved in [47] by deriving and applying new formulas for value functions in parametric optimization, which allow us to fully *avoid convexification* in the necessary optimality conditions established in [12]. In particular, under the same assumptions as in [12, Theorem 3.1] with the upper-level constraint set Ω in (35) described by the inequalities

$$\Omega := \left\{x \in \mathbb{R}^n \,\middle|\, g_j(x) \le 0,\;\; j = 1,\ldots,p\right\} \tag{37}$$

involving the smooth initial data φ, ψ, f_i, and g_j in (35)–(37), we get the following necessary conditions for a local optimal solution $(\bar{x},\bar{y})$ to the bilevel program under consideration: there are $\gamma > 0$ and nonnegative multipliers $\lambda_1,\ldots,\lambda_m$, $\alpha_1,\ldots,\alpha_m$, and $\beta_1,\ldots,\beta_p$ such that

$$\begin{aligned}
&\nabla_x\varphi(\bar{x},\bar{y}) + \sum_{i=1}^m(\alpha_i - \gamma\lambda_i)\nabla_x f_i(\bar{x},\bar{y}) + \sum_{j=1}^p \beta_j \nabla g_x(\bar{x}) = 0,\\
&\nabla_y\varphi(\bar{x},\bar{y}) + \gamma\nabla_y\psi(\bar{x},\bar{y}) + \sum_{i=1}^m \alpha_i \nabla f_y(\bar{x},\bar{y}) = 0,\\
&\nabla_y\psi(\bar{x},\bar{y}) + \sum_{i=1}^m \lambda_i \nabla_y f_i(\bar{x},\bar{y}) = 0,\\
&\lambda_i f_i(\bar{x},\bar{y}) = 0,\; \alpha_i f_i(\bar{x},\bar{y}) = 0 \;\text{ for }\; i = 1,\ldots,m,\; \beta_j g_j(\bar{x}) = 0 \;\text{ for }\; j = 1,\ldots,p.
\end{aligned} \tag{38}$$

In [12, 47], the reader can find more results and discussions on bilevel programs with nonsmooth data, and also with fully convex and linear structures.

6 Multiobjective Optimization and Equilibria

It is difficult to overstate the importance of multiobjective optimization and related equilibrium problems for both optimization/equilibrium theory and practical applications; see, e.g., [6, 7, 8, 17, 19, 21, 22, 32, 56, 61, 63] with the discussions and references therein. It has been well recognized that the advanced methods of variational analysis and generalized differentiation provide useful tools for the study of such problems and lead to significant progress in the theory and applications. In this section we discuss some latest advances in this direction based mostly on the recent research by the author and his collaborators.

A large class of constrained *multiobjective optimization* problems is described as:

$$\text{minimize } F(x) \text{ subject to } x \in \Omega \subset X, \tag{39}$$

where the *cost mapping* $F\colon X \rightrightarrows Z$ is generally *set-valued*, and where "minimization" is understood with respect to some *partial ordering* on Z. Thus (39) is a problem of *set-valued optimization*, while the term of *vector optimization* is usually used when $F = f\colon X \to Z$ is a single-valued mapping. We prefer to unify both set-valued and vector optimization problems under the name of multiobjective optimization. It is well known that various notions of *equilibrium* can be written in (or reduce to) form (39).

In [32, Sect. 5.3] and in the subsequent papers [2, 37, 40] we paid the main attention to the study of *generalized order optimality* defined as follows: given an ordering set $\Theta \subset Z$ with $0 \in \Theta$, we say tat $\bar{x} \in \Omega$ is a *locally* (f,Θ,Ω)*-optimal* if there are a neighborhood U of $\bar{x}$ and a sequence $\{z_k\} \subset Z$ with $\|z_k\| \to 0$ as $k \to \infty$ such that

$$f(x) - f(\bar{x}) \notin \Theta - z_k \text{ for all } x \in \Omega \cap U, \ k \in \mathbb{N}. \tag{40}$$

The (generally nonconvex and nonconical) set Θ in (40) can be viewed as a generator of an extended *order/preference relation* on Z and encompasses standard notions of multiobjective optimization and equilibria. In fact the above notion of generalized order optimality is induced by the notion of *local extremal points* of sets discussed in Sect. 3; see [32, Subsect. 5.3.1] for more details and examples.

The main results of [2, 37, 40] provide *necessary optimality conditions* for multiobjective problems with respect to the above generalized order optimality under various constraints (geometric, functional, operator, equilibrium, and their specifications) in finite and infinite dimensions. The results obtained are expressed via the robust/limiting generalized differential constructions discussed in Sect. 2. In [32, Subsect. 5.4.2] and [36], pointbased *necessary and sufficient* conditions are derived for *linearly suboptimal solutions* to multiobjective problems generated by linear subextremality of sets considered in Sect. 3.

Paper [51] is devoted to the study and applications of a remarkable and rather new class of *equilibrium problems with equilibrium constraints* (EPECs), which can be treated as *hierarchical games* defined by some equilibrium notions on both lower and upper levels of hierarchy. In [51], we pay a particular attention to the case of *weak Pareto optimality/equilibrium* on the upper level and *mixed complementarity constraints* on the lower level. Such problems can be modeled in the above framework of *multiobjective optimization with equilibrium constraints*. The necessary optimality conditions derived in [51] are based on the robust generalized differentiation constructions of Sect. 2, while they are finally presented fully in terms of the initial data and used in developing and implementing *numerical techniques*. The applications given in [51] concern *oligopolistic market* models that primarily motivate the research.

Paper [39] concerns a thorough study of multiobjective optimization problems with equilibrium constraints, where the notion of optimality is generated by *closed preference relations*. Given a subset $\Xi \subset Z \times Z$, we define the *preference* $\prec$ on Z by

$$z_1 \prec z_2 \text{ if and only if } (z_1, z_2) \in \Xi \tag{41}$$

and say that $\prec$ is *locally closed* around $\bar{z}$ if there is a neighborhood U of $\bar{z}$ such that:

(a) preference $\prec$ is *nonreflexive*, i.e., $(z,z) \notin \Xi$;
(b) preference $\prec$ is *locally satiated* around $\bar{z}$, i.e., $z \in \operatorname{cl} \mathscr{L}(z)$ for all $z \in U$, where the *level set* $\mathscr{L}(z)$ corresponding to $\prec$ is defined by

$$\mathscr{L}(z) := \big\{ u \in Z \big|\, u \prec z \big\}; \tag{42}$$

(c) preference $\prec$ is *almost transitive* on Z, i.e.

$$v \prec z \text{ whenever } v \in \operatorname{cl} \mathscr{L}(u), \;\; u \prec z, \;\text{ and } \; v, z, u \in U. \tag{43}$$

Observe that ordering relations on Z given by the generalized order optimality as in (40) and by closed preferences in (43) are generally independent. In particular, the almost transitivity of a *Pareto-type* preference given by

$$z_1 \prec z_2 \text{ if and only if } z_2 - z_1 \in \Theta \tag{44}$$

via a closed cone $\Theta \subset Z$ is *equivalent* to the *convexity* and *pointedness* of the cone Θ, which means that $\Theta \cap (-\Theta) = \{0\}$. The latter is not required in (40) and does not hold in fact for a number of useful preferences important in the theory and applications, e.g., for the *lexicographical ordering* on $\mathbb{R}^n$; see [32, Subsect. 5.3.1] for more details and discussions.

Note that the necessary optimality conditions obtained in [39] for multiobjective problems described via closed preferences employ the notion of the *extended normal cone* to parameterized/moving sets $\Omega(\cdot)$ defined by

$$N_+\big(\bar{x}; \Omega(\bar{z})\big) := \mathop{\operatorname{Lim\,sup}}_{(z,x) \overset{\operatorname{gph}\Omega}{\to} (\bar{z},\bar{x})} \hat{N}\big(x, \Omega(z)\big) \;\text{ at }\; \bar{x} \in \Omega(\bar{z}). \tag{45}$$

We refer the reader to the recent paper [54] for a comprehensive study of the extended normal cone (45) and associated coderivative and subdifferential constructions for moving objects (calculus rules, various relationships, normal compactness properties, etc.). In [39], the extended normal cone construction (45) is applied to express a part of necessary optimality conditions related to the moving level sets (42).

The main focus of [3] is on the study of the constrained multiobjective optimization problems (39) with general set-valued costs. We consider there two classical notions of minimizers/equilibria: *Pareto* and *weak Pareto*. The first notion corresponds to the preference on Z given by a closed and convex cone $\Theta \subset Z$ (which is assumed to be pointed in [3]), while the weak one assumes in addition that $\operatorname{int}\Theta \neq \emptyset$. Although the latter is a serious restriction, the vast majority of publications on multiobjective optimization, even in the simplest frameworks, concern *weak* Pareto minimizers, which are much more convenient to deal with in the vein of the conventional *scalarization* techniques.

In [3], we derive necessary conditions for both Pareto and weak Pareto minimizers in terms of our coderivatives discussed in Sect. 2 and also using new *subdifferential* constructions for *set-valued mappings* with values in *partially ordered spaces* that are extensions of those in (9)–(11) to the case of vector-valued and set-valued mappings. The basic techniques of [3] involves new versions of *variational principles* that are set/vector-valued counterparts of the classical Ekeland variational principle [16] and the subdifferential variational principle given in [31, Subsect. 2.3.2]. Furthermore, paper [3] contains new *existence theorems* for optimal solutions to (39) that employ, in particular, the following *subdifferential Palais-Smale condition* expressed in terms of the aforementioned analog of the basic subdifferential (10) for set/vector-valued mappings with values in partially ordered spaces: every sequence $\{x_k\} \subset X$ such that

$$\text{there are } z_k \in F(x_k) \text{ and } x_k^* \in \partial F(x_k, z_k) \text{ with } \|x_k^*\| \to 0 \text{ as } k \to \infty \tag{46}$$

contains a convergent subsequence, provided that $\{z_k\}$ is (quasi)bounded from below.

In [4], we obtain a number of extensions of the existence theorems and necessary optimality conditions from [3] to multiobjective problems with various *constraints*, including those of the *equilibrium* type. This becomes possible due to the availability of coderivative/subdifferential *calculus* for the generalized differential constructions used in [3, 4] (including the aforementioned new subdifferentials for set/vector-valued mappings), which particularly allows us to deal with various constraint structures.

Paper [6] addresses the study of the new notions of *relative Pareto minimizers* to constrained multiobjective problems that are defined via several kinds of *relative interiors* of ordering cones and occupy intermediate positions between the classical notions of Pareto and weak Pareto efficiency/optimality in finite-dimensional and infinite-dimensional spaces. Using advanced tools of variational analysis and generalized differentiation, we establish the *existence* of *relative Pareto minimizers* to general multiobjective problems under a *refined version of the subdifferential Palais-Smale condition* for set-valued mappings with values in partially ordered spaces and then derive *necessary optimality conditions* for these minimizers (as well as for conventional efficient and weak efficient counterparts) that are new in both finite-dimensional and infinite-dimensional settings. The proofs in [6] are mainly based on *variational and extremal principles* of variational analysis including certain new versions of them derived in the paper.

Finally in this section, we mention the recent developments in [5] devoted to so-called *super minimizers* to multiobjective optimization problems (39) with generally set-valued cost mappings. This notion is induced by the concept of *super efficiency* introduced in [8], which refines and/or unifies various modifications of *proper efficiency* and reflects crucial features of solutions to vector optimization problems important from the viewpoints of both the theory and applications. We derive necessary conditions for super minimizers using advanced tools of variational analysis and generalized differentiation that are new in both finite-dimensional and infinite-dimensional settings for problems with single-valued and set-valued objectives. The results obtained are expressed in generally independent *coderivative* and *subdifferential* forms. Then a part of [5] concerns establishing relationships between these notions for *set/vector-valued* mappings with values in *partially ordered* spaces, which are also important for further developments and applications.

7 Metric Regularity and Lipschitzian Stability of Parametric Variational Systems

It has been well recognized that the property of set-valued mappings known as *metric regularity*, as well as the *linear openness/covering* property equivalent to it, play an important role in many aspects of nonlinear and variational analysis and their applications; see, e.g., [7, 14, 18, 20, 23, 24, 31, 32, 58] with the extensive bibliographies therein. In the aforementioned references, the reader can find verifiable conditions ensuring these properties and their implementations in specific situations mainly related to the *implicit functions* and *multifunctions* frameworks and to the so-called *parametric constraint systems* in nonlinear analysis and optimization. The latter class of systems incorporates, in particular, sets of *feasible* solutions to various constrained optimization and equilibrium problems.

Recall that $F\colon X \rightrightarrows Y$ is *metrically regular* around $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and a number $\mu > 0$ such that

$$\operatorname{dist}\big(x;F^{-1}(y)\big) \le \mu \operatorname{dist}\big(y;F(x)\big) \ \text{ whenever } \ x \in U \ \text{ and } \ y \in V. \tag{47}$$

Further, we say that $F\colon X \rightrightarrows Y$ is *Lipschitz-like* around $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and a number $\ell \ge 0$ such that

$$F(x) \cap V \subset F(u) + \ell\|x-u\|\mathbb{B} \ \text{ for all } \ x,u \in U. \tag{48}$$

The latter property is also known as the Aubin "pseudo-Lipschitz" property of set-valued mappings; see [31, 58]. When $V = Y$ in (48), it reduces to the classical (Hausdorff) *local Lipschitzian* property of F around $\bar{x} \in \operatorname{dom} F$. Note that the Lipschitzian properties under consideration are *robust*, i.e., stable with respect to small perturbations of the initial data.

It is well known and utilized in nonlinear and variational analysis that the metric regularity property of F around $(\bar{x},\bar{y})$ is *equivalent* to the Lipschitz-like property of its *inverse* around $(\bar{y},\bar{x})$ with the same modulus in (47) and (48). Similar relationships hold true for certain *semilocal* and *global* modifications of the above *local* metric regularity and Lipschitzian properties and their *linear openness/covering* counterparts; see, [31, Sect. 1.2] for more details and discussions.

Observe that both metric regularity and Lipschitzian properties are defined in *primal* spaces and are *derivative-free*, i.e., they do not depend on any derivative-like construction. It turns out nevertheless that, due to *variational/extremal principles*, they admit *complete dual-space characterizations* in both finite-dimensional and infinite-dimensional spaces via appropriate *coderivatives* of set-valued mappings; see [29], [31, Chapter 4], and [58, Chapter 9] with comprehensive references and commentaries.

We have discussed in Sect. 4 a significant role of *parametric variational systems* of the types considered therein in variational analysis, optimization/equilibrium theory, and their numerous applications. It is shown in many publications that *robust Lipschitzian properties* are *intrinsic* for such systems being fulfilled under natural assumptions; see, e.g., the recent developments in [31, Sect. 4.4] and [33] based on *coderivative analysis* that largely revolves around the *Fredholm qualification condition* (33). It surprisingly happens, however, that it is *not the case for metric regularity* and the equivalent properties of linear openness/covering, which fail to be fulfilled for major classes of parametric variational systems.

In what follows, we present some results in this direction recently obtained in [43]. They are largely based on the *equivalence* [18] between metric regularity of the solution maps in systems (23), (26), and (27) and the Lipschitz-like property of the field/subdifferential mappings in these systems under the assumptions made. The latter property does not hold in the major cases under considerations; see [15, 26, 43] for more details.

Theorem 2. (Failure of metric regularity for generalized equations with monotone fields). *Let $f\colon X \times Y \to Y^*$ be a mapping between Asplund spaces that is strictly differentiable at $(\bar{x},\bar{y})$ with the surjective partial derivative $\nabla_x f(\bar{x},\bar{y})$, and let $Q\colon Y \rightrightarrows Y^*$ be locally closed-graph around $(\bar{y},\bar{y}^*)$ with $\bar{y}^* := -f(\bar{x},\bar{y}) \in Q(\bar{y})$. Assume in addition that Q is monotone and that there is no neighborhood of $\bar{y}$ on which Q is single-valued. Then the solution map $S\colon X \rightrightarrows Y$ in* (23) *with $Q = Q(y)$ is not metrically regular around $(\bar{x},\bar{y})$.*

Since the *set-valuedness* of field mappings is a *characteristic* feature of *generalized* equations as a satisfactory model to describe *variational systems* (otherwise they reduce just to standard equations, which are not of particular interest in the variational framework under consideration), the conclusion of Theorem 2 reads that parametric variational systems with *monotone fields* are *not metrically regular* under the strict differentiability and surjectivity assumptions on base mappings, which do not seem to be restrictive. A major consequence of Theorem 2 is the following corollary concerning *subdifferential* systems with *convex* potentials, which encompass the classical cases of variational inequalities and complementarity problems in (24) that correspond to the *highly nonsmooth* (extended-real-valued) case of the convex *indicator functions* $\varphi(y) = \delta(y;\Omega)$ in (23) with $Q(y) = \partial\varphi(y)$.

Corollary 1. (Failure of metric regularity for subdifferential variational systems with convex potentials). *Let $Q(y) = \partial\varphi(y)$ in* (23)*, where $f\colon X \times Y \to Y^*$ is a mapping between Asplund spaces that is strictly differentiable at $(\bar{x},\bar{y})$ with the surjective partial derivative $\nabla_x f(\bar{x},\bar{y})$, and where $\varphi\colon Y \to \bar{\mathbb{R}}$ is a l.s.c. convex function finite at $\bar{y}$ and such that there is no neighborhood of $\bar{y}$ on which φ is Gâteaux differentiable. Then the solution map S in* (23) *is not metrically regular around $(\bar{x},\bar{y})$.*

In fact, essentially more general *composite* subdifferential structures of parametric variational systems prevent the fulfillment of metric regularity for solutions maps with *no* reduction to the field monotonicity. In particular, it is proved in [43, Theorem 5.3 and 5.4] that *metric regularity fails* for the composite subdifferential systems (26) and (27) in Asplund spaces with $g = g(y)$ provided that f satisfies the assumptions of Theorem 2, that g is continuously differentiable around $\bar{y}$ in (27) while twice continuously differentiable around $\bar{y}$ in (26) with the surjective derivative $\nabla g(\bar{y})$ in both cases, and that ψ is l.s.c., convex, and *not* Gâteaux differentiable around the point $g(\bar{y})$.

In the case of *Hilbert spaces*, the results of Corollary 1 and the aforementioned ones for the composite structures (26) and (27) can be extended to subdifferential variational systems generated by essentially larger (than convex) classes of extended-real-valued functions. Recall [58] that $\varphi\colon X \to \bar{\mathbb{R}}$ is *subdifferentially continuous* at $\bar{x}$ for some subgradient $\bar{x}^* \in \partial\varphi(\bar{x})$ if $\varphi(x_k) \to \varphi(\bar{x})$ whenever $x_k \to \bar{x}$, $x_k^* \xrightarrow{w^*} \bar{x}^*$ as $k \to \infty$ with $x_k^* \in \partial\varphi(x_k)$ for all $k \in \mathbb{N}$. Further, φ is *prox-regular* at $\bar{x} \in \operatorname{dom}\varphi$ for some $\bar{x}^* \in \partial\varphi(\bar{x})$ if it is l.s.c. around $\bar{x}$ and there are $\gamma > 0$ and $\eta \geq 0$ such that

$$\begin{aligned}&\varphi(u) \geq \varphi(x) + \langle \bar{x}^*, u - x\rangle - \frac{\eta}{2}\|u - x\|^2 \ \text{ for all } \ x^* \in \partial\varphi(x)\\ &\quad \text{with } \ \|x^* - \bar{x}^*\| \leq \gamma,\ \|u - \bar{x}\| \leq \gamma,\ \|x - \bar{x}\| \leq \gamma, \ \text{ and } \ \varphi(x) \leq \varphi(\bar{x}) + \gamma.\end{aligned} \tag{49}$$

Both properties above hold for broad classes of functions important in variational analysis and optimization. This is the case, in particular, for the so-called *strongly amenable* functions; see [58] and also [31, 32] for more details, references, and applications.

Theorem 3. (Failure of metric regularity for composite subdifferential variational systems with prox-regular potentials). *Let $(\bar{x},\bar{y}) \in \operatorname{gph} S$ for S given in* (26)*, where $g\colon Y \to W$ is twice continuously differentiable around $\bar{y}$ with the surjective derivative $\nabla g(\bar{y})$, where $f\colon X \times Y \to Y^*$ is strictly differentiable at $(\bar{x},\bar{y})$ with the surjective partial derivative $\nabla_x f(\bar{x},\bar{y})$, where the spaces X, Y, and Y^* are Asplund while W is Hilbert. Set $\bar{w} := g(\bar{y})$ and assume in addition that:*

(i) either ψ is locally Lipschitzian around $\bar{w}$;
(ii) or ψ is prox-regular and subdifferential continuous at $\bar{w}$ for the basic subgradient $\bar{v} \in \partial\psi(\bar{w})$, which is uniquely determined by $\nabla g(\bar{y})^*\bar{v} = -f(\bar{x},\bar{y})$.

Then S is not metrically regular around $(\bar{x},\bar{y})$ provided that there is no neighborhood of $\bar{w}$ on which ψ is Gâteaux differentiable.

Theorem 4. (Failure of metric regularity for subdifferential variational systems with composite fields and prox-regular potentials). *Let $(\bar{x},\bar{y}) \in \operatorname{gph} S$ for S defined by* (27)*, where $g\colon Y \to W$ is strictly differentiable at $\bar{y}$ with the surjective derivative $\nabla g(\bar{y})$, where $f\colon X \times Y \to W$ is strictly differentiable at $(\bar{x},\bar{y})$ with the surjective partial derivative $\nabla_x f(\bar{x},\bar{y})$, where the spaces X and Y are Asplund while W is Hilbert. Set $\bar{w} := g(\bar{y})$ and assume in addition that either* (i) *or* (ii) *of Theorem* 3 *is satisfied, and that there is no neighborhood of $\bar{w}$ on which ψ is Gâteaux differentiable. Then the solution map S is not metrically regular around $(\bar{x},\bar{y})$.*

In [43], the reader can find the proofs of these theorems and more discussions on them and related results for metric regularity and Lipschitzian stability of variational systems.

8 Optimal Control of Constrained Evolution Inclusions with Discrete and Continuous Time

As discussed in Sect. 1, problems of *optimal control* and related problems of *dynamic optimization* have always been among the strongest motivations and most important areas for applications of advanced methods and constructions of modern variational analysis and generalized differentiation. In this section we briefly review recent results on optimal control and related problems obtained by the author and his collaborators in [13, 34, 35, 52, 55].

In [35], we study the following problem of dynamic optimization governed by *discrete-time inclusions* with endpoint constraints of inequality, equality, and geometric types:

$$\begin{cases} \text{minimize } \; \varphi_0(x_0,x_K) \; \text{ subject to } (x_0,x_K) \in \Omega, \\ x_{j+1} \in F_j(x_j), \; j = 0,\ldots,K-1, \\ \varphi_i(x_0,x_K) \le 0, \; i = 1,\ldots,m, \; \varphi_i(x_0,x_K) = 0, \; i = m+1,\ldots,m+r, \end{cases} \tag{50}$$

where $F_j\colon X \rightrightarrows X$, $\varphi_i\colon X^2 \to \mathbb{R}$, $\Omega \subset X^*$ and $K \in \mathbb{N}$. Observe that the inclusion model in (50) encompasses more conventional *discrete control systems* of the parameterized type

$$x_{j+1} = f_j(x_j,u_j), \; u_j \in U_j \; \text{ as } \; j = 0,\ldots,K-1 \tag{51}$$

with *explicit* control variables u_j taking values in some admissible control regions U_j.

The following major result is established in [35] based on the reduction to the *Lagrange principle* for non-dynamic constrained optimization problems discussed at the end of Sect. 3 and then on employing appropriate rules of *generalized differential and SNC calculi*.

Theorem 5. (Extended Euler-Lagrange conditions for discrete optimal control). *Let $\{\bar{x}_j|\ j=0,\ldots,K\}$ be a local optimal solution to the discrete optimal control problem* (50)*. Assume that X is Asplund, that φ_i are locally Lipschitzian around $(\bar{x}_0,\bar{x}_K)$ for all $i=0,\ldots,m+r$ while Ω is locally closed around this point, and that the graphs of F_j are locally closed around $(\bar{x}_j,\bar{x}_{j+1})$ for every $j=0,\ldots,K-1$. Suppose also that all but one of the sets Ω and $\operatorname{gph} F_j$, $j=0,\ldots,K-1$, are SNC at the points $(\bar{x}_0,\bar{x}_K)$ and $(\bar{x}_j,\bar{x}_{j+1})$, respectively. Then there are multipliers $(\lambda_0,\ldots,\lambda_{m+r})$ and an adjoint discrete trajectory $\{p_j\in X^*|\ j=0,\ldots,K\}$, not all zero, satisfying the relationships:*

- *the Euler-Lagrange inclusion*

$$-p_j\in D^*_N F_j(\bar{x}_j,\bar{x}_{j+1})(-p_{j+1})\ \textit{ for }\ j=0,\ldots,K-1, \tag{52}$$

- *the transversality inclusion*

$$(p_0,-p_K)\in\partial\Big(\sum_{i=0}^{m+r}\lambda_i\varphi_i\Big)(\bar{x}_0,\bar{x}_K)+N\big((\bar{x}_0,\bar{x}_K);\Omega\big), \tag{53}$$

- *the sign and complementary slackness conditions*

$$\lambda_i\ge 0\ \textit{ for }\ i=0,\ldots,m,\ \lambda_i\varphi_i(\bar{x}_0,\bar{x}_K)=0\ \textit{ for }\ i=1,\ldots,m. \tag{54}$$

Note that if F_j is inner/lower semicontinuous at $(\bar{x}_j,\bar{x}_{j+1})$ and *convex-valued* around these points for all $j=0,\ldots,K-1$, then the Euler-Lagrange inclusion (52) implies the relationships of the *discrete maximum principle*:

$$\langle p_{j+1},\bar{x}_{j+1}\rangle=\max_{v\in F(\bar{x}_j)}\langle p_{j+1},v\rangle\ \text{ for all }\ j=0,\ldots,K-1, \tag{55}$$

which provide necessary optimality conditions for problem (50) along with (52)–(54).

Observe that the results of Theorem 5 allow us to establish necessary optimality conditions (52)–(54) and the maximum principle (55) with *no SNC* (or *finite codimensionality*, or *interiority*) assumptions imposed on the endpoint constraint/target set Ω and to cover, e.g., the classical two-point constraint case in (50) that has always been an obstacle in infinite-dimensional optimal control, including that for smooth systems (51).

By using generalized differential and SNC calculus rules, Theorem 5 induces the corresponding necessary optimality conditions for optimal control problems of constrained *parametric discrete-time evolution inclusions* of the type

$$x_{j+1}\in x_j+hF_j(x_j),\ j=0,\ldots,K-1. \tag{56}$$

It is worth mentioning that explicit control counterparts as in (51) of the parametric discrete-time systems (56), considered as a *process* with $h \downarrow 0$, possess a number of important specific features that are *not* inherent in general parametric discrete systems with fixed parameters h. An especially remarkable fact for optimal control of such systems with *smooth* velocity mappings f_j is the validity of necessary optimality conditions in the form of the *approximate maximum principle* with *no convexity* requirements. The approximate maximum condition means that the exact one as in (55) is replaced by its $\varepsilon(h)$-*perturbation* with $\varepsilon(h) \to 0$ as $h \downarrow 0$; see [32, Sect. 6.4] for more details, references, and commentaries.

Systems of type (56) arise, in particular, from *discrete/finite-difference approximations* of *continuous-time* evolution systems governed by *differential inclusions*

$$\dot{x}(t) \in F\big(x(t),t\big),\ x \in X,\ \text{ a.e. } t \in [a,b]. \tag{57}$$

In fact, the approach to the study of continuous-time systems of type (57) and optimization problems for them via *well-posed discrete approximations* has been among the author's main interests and developments for a long time; see, e.g., [30], [32, Chapter 5] with the references and commentaries therein. The *major steps* of this approach to derive necessary optimality conditions for various constrained optimal control problems governed by continuous-time systems are as follows:

(a) To construct a *well-posed* sequence of discrete-time problems that *approximate* in an *appropriate sense* the original continuous-time problem of dynamic optimization.
(b) To derive *necessary optimality* conditions for the approximating *discrete-time* problems by reducing them to non-dynamic problems of mathematical programming and employing then *generalized differential calculus*.
(c) By *passing to the limit* in the obtained results for discrete approximations to establish necessary conditions for the *given optimal solution* to the original problem.

Note that each of the above steps in the study of relationships between continuous-time systems and their discrete approximations is certainly of its own interests regardless of deriving necessary optimality conditions for the continuous-time dynamics. In particular, step (a) and its modifications are important for *numerical analysis* of continuous-time systems.

In this vein, paper [52] deals with establishing the *epi-convergence* of discrete approximations to the so-called *generalized Bolza problem* of dynamic optimization, which encompasses a number of the most interesting optimal control problems governed by differential inclusions of type (57) with *finite-dimensional* state spaces $X = \mathbb{R}^n$. The methods developed in this study and the results obtained seem to be suitable for extensions to *higher dimensions* (versus $t \in \mathbb{R}$) in the framework of *finite element methods*.

Paper [13] also goes in the direction of the aforementioned step (a) and is devoted to the study of well-posedness of discrete approximations to *nonconvex* differential inclusions of type (57) with *Hilbert* state spaces X. The underlying feature of the problems under consideration in [13] is a *one-sided Lipschitz* condition imposed on

$F(\cdot,t)$, which is a significant improvement of the conventional Lipschitz continuity studied in prior publications. Among the main results of [13] we mention establishing efficient conditions that ensure the *strong approximation* (in the $W^{1,p}$-norm as $p \geq 1$) of feasible trajectories for one-sided Lipschitzian differential inclusions by those for their discrete approximations and also the *strong convergence* of optimal solutions to the corresponding dynamic optimization problems under discrete approximations. To proceed with the latter issue, we derive a new extension of the Bogolyubov-type *relaxation/density* theorem to the case of differential inclusions satisfying the modified one-sided Lipschitzian condition. All the results obtained are new not only in the infinite-dimensional Hilbert space framework but also in finite-dimensional spaces.

Paper [34] develops *all the three* of the aforementioned steps (a)–(c) in the implementation of the *method of discrete approximations* to derive new necessary optimality conditions for *nonconvex evolution/differential inclusions* of type (57) in the case of *Asplund* state spaces X. Dynamic optimization problems (of the Bolza and Mayer types) are considered in [34] subject to *finitely many* of the Lipschitzian *endpoint constraints*

$$\varphi_i\big(x(b)\big) \leq 0,\ i=1,\ldots,m,\ \varphi_i\big(x(b)\big)=0,\ i=m+1,\ldots,m+r, \tag{58}$$

on the trajectories for the evolution inclusion (57) with $x(a)=x_0$. The optimality conditions derived in [34] do *not impose* any *SNC/finite codimension* requirements on the target sets in (58) in contrast to geometric endpoint constraints of the type $x(b)\in\Omega$ studied previously in the author's book [32, Sect. 6.1 and 6.2]. The continuous-time counterpart of the extended *Euler-Lagrange inclusion* obtained in [34] is given by

$$\dot{p}(t) \in \mathrm{clco}\, D^*_N F\big(\bar{x}(t),\dot{\bar{x}}(t)\big)\big(-p(t)\big)\ \text{ a.e. } t\in[a,b] \tag{59}$$

together with the corresponding transversality, sign, complementary slackness, and maximum conditions as in (53)–(55). Note that, in contrast to the discrete case of (52), the Euler-Lagrange inclusion (59) involves the *convexification* of the coderivative values, while the *maximum condition*

$$\big\langle p(t),\dot{\bar{x}}(t)\big\rangle = \max_{v\in F(\bar{x}(t))}\big\langle p(t),v\big\rangle\ \text{ a.e. } t\in[a,b] \tag{60}$$

does *not* require *any convexification*. The latter is due the "hidden convexity" property (of the Lyapunov-Aumann type), which is automatically generated by the continuous-time dynamics; see [32, 34] for more results and discussions in this direction.

Finally in this section, we mention new results on the well-posedness of discrete approximations and necessary optimality conditions obtained in [55] for dynamic optimization problems governed by constrained *delay-differential inclusions* of the type

$$\begin{cases} \dot{x}(t) \in F(x(t), x(t-\Delta), t) \ \text{ a.e. } \ t \in [a,b], \\ x(t) \in C(t) \ \text{ a.e. } \ t \in [a-\Delta, a), \quad \Delta > 0, \\ (x(a), x(b)) \in \Omega \subset X^2 \end{cases} \tag{61}$$

with an *Asplund* state space X. A specific feature of the delay system (61), which does not have any analogs for nondelayed systems, is the presence of *set-valued initial conditions* of the time $x(t) \in C(t)$ on $[a-\Delta, a)$, which particularly provides an additional source for optimization. The results obtained in [55] develop and extend those from [32, 34] for the delay-differential problems under consideration, with deriving appropriate delay counterparts of conditions (59) and (60) as well as the new one corresponding to the multivalued "initial tail" part on $[a-\Delta, a)$.

9 Feedback Control of Constrained Parabolic Systems in Uncertainty Conditions

In the concluding section of the paper we discuss recent results by the author on *optimal control* and *feedback design* of *state-constrained parabolic systems* in *uncertainty conditions*. Problems of this type are among the most challenging and difficult in dynamic optimization for any kind of dynamical systems. The feedback design problem is formulated in the *minimax sense* to ensure *stabilization* of transients within the prescribed diapason and *robust stability* of the closed-loop control system under all feasible perturbations with *minimizing* an integral cost functional in the *worst* perturbation case.

The original motivation for our developments comes from practical design problems of automatic control of the soil groundwater regime in irrigation engineering networks functioning under uncertain weather and environmental conditions. In [41, 42, 44], we study such problems for parabolic systems with controls acting in boundary conditions of various types (Dirichlet, Neumann, Robin/mixed). In what follows we present the problem formulation and discuss the major results for the case of *Dirichlet boundary conditions*, which offer the *least regularity* properties for the parabolic dynamics and appear to be the *most challenging* in control theory for parabolic systems.

The system dynamics in the problem under consideration is given by the multidimensional *linear parabolic equation*

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = w(t) \quad \text{a.e. in } \ Q := [0,T] \times \Omega, \\ y(0,x) = 0, \quad x \in \Omega, \\ y(t,x) = u(t), \quad (t,x) \in \Sigma := [0,T] \times \Gamma \end{cases} \tag{62}$$

with *controls* $u(\cdot)$ acting in the Dirichlet boundary conditions and distributed *perturbations* $w(\cdot)$ in the right-hand side of the parabolic equation. In (62), A is a *self-adjoint* and *uniformly strongly elliptic operator* on $L^2(\Omega)$ defined by

$$Ay := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big) - cy, \tag{63}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with the the boundary Γ that is supposed to be a sufficiently smooth $(n-1)$-dimensional manifold, and where $T > 0$ is a fixed time bound.

The sets of *admissible controls* U and *admissible perturbations* W are given by

$$U := \Big\{u \in L^\infty[0,T]\Big| -\alpha \le u(t) \le \alpha \text{ a.e. } t \in [0,T]\Big\}, \tag{64}$$

$$W := \Big\{w \in L^\infty[0,T]\Big| -\beta \le w(t) \le \beta \text{ a.e. } t \in [0,T]\Big\} \tag{65}$$

with some fixed bounds $\alpha, \beta > 0$ in the *pointwise/magnitude* constraints (64) and (65).

The underlying requirement on the system performance is to *stabilize* transients $y(t,x_0)$ near the initial equilibrium state $y(x,0) \equiv 0$ with a given accuracy $\eta > 0$ during the whole dynamic process. This is formalized via the *pointwise state constraints*

$$-\eta \le y(t,x_0) \le \eta \text{ a.e. } t \in [0,T]. \tag{66}$$

A characteristic feature of the dynamical process described by (62) is the *uncertainty* of perturbations $w \in W$: we can operate only with the bound β of the admissible region (65). Thus we can keep the system transients $y(t,x_0)$ within the prescribed stabilization region (66) only by using *feedback* boundary controls $u(\cdot)$ depending on the current state position $\xi = y(t,x_0)$ for each $t \in [0,T]$.

To formalize this description, consider a function $f\colon \mathbb{R} \to \mathbb{R}$ and construct boundary controls in (62) via the *feedback law*

$$u(t) := f\big(y(t,x_0)\big),\ t \in [0,T], \tag{67}$$

which defines a *feasible feedback regulator* if it generates controls $u(t)$ by (67) belonging to the admissible set U from (64) and keeps the corresponding transients $y(t,x_0)$ of (62) within the constraint area (66) for every admissible perturbation $w \in W$ from (65). We estimate the quality of feasible regulators $f = f(\xi)$ by the (energy-type) *cost functional*

$$J(f) := \max_{w \in W}\Big\{\int_0^T \big|f\big(y(t,x_0)\big)\big|\,dt\Big\}. \tag{68}$$

The *maximum* operation in (68) reflects the required control energy needed to neutralize the adverse effect of the *worst perturbations* from (65) and to keep the state performance within the prescribed area (66). Finally, denote by $\mathscr{F}$ the set of all feasible feedback regulators and formulate the *minimax feedback control problem* as follows:

$$\text{minimize } J(f) \text{ over } f \in \mathscr{F}. \tag{69}$$

It has been well recognized in control theory and applications that *feedback* control problems are the most challenging and important for any type of dynamical systems, while PDE systems provide additional difficulties and much less investigated in comparison with the ODE dynamics. Furthermore, significant complications come from *pointwise/hard state constraints*, which are of high nontriviality even for open-loop control problems. We are not familiar with any constructive device applicable to the feedback control problem (P) under consideration among a variety of approaches and results available in the theories of differential games, H_∞-control, Riccati's feedback synthesis, and other developments in general settings; see more discussions and references in the aforementioned papers.

In these papers, we develop an approach to solving the feedback control problem (69), which is essentially based on certain underlying features of the parabolic dynamics, particularly on the *monotonicity property* of transients that is eventually related to the fundamental *Maximum Principle* for parabolic equations. Due to this property and the specific structures of the cost functional (68) and boundary controls in (62) and by employing the *convolution representation* of the transients obtained [53], we are able to select the *worst perturbations* in the area (65) for the class of *nonincreasing* and *odd feedbacks* (67). This allows us to study the corresponding *open-loop* optimal control problem with *pointwise state constraints* as a reaction of the parabolic system to the worst perturbations. Using the *spectral* Fourier-type representation of solutions to the parabolic system (62) and assuming the *positivity* of the *first eigenvalue* of the elliptic operator A in (63)—which is often the case— we observe the *dominance* of the *first term* in the exponential series representation of solutions to (62) as $t \to \infty$. In this way, we justify an efficient approximation of the open-loop optimal control problem for the parabolic system under consideration by that for the corresponding *ODE system* with state constraints on a sufficiently *large* time interval. Moreover, the approximating ODE optimal control problem is solved *exactly* by constructing *yet another approximation* of state constraints, employing the *Pontryagin maximum principle* that provides *necessary and sufficient* optimality conditions for the unconstrained approximating problems with both *bang-bang* and *singular modes* of optimal controls, and then by passing to the limit while meeting the state constraints. Furthermore, the *state constraints* occur to be a *regularization factor*, which simplifies the structure of optimal controls, especially when the time interval becomes bigger and bigger; this reveals the fundamental *turnpike property* of such dynamic systems expanding to the *infinite horizon*.

Thus using the ODE approximation described above, we justify an easily implemented *suboptimal* (or *near-optimal*) *structures* of optimal controls in both *open-loop* and *closed-loop* modes and then *optimize their parameters* along the *parabolic dynamics*. This allows us to arrive at a *three-positional feedback regulator* $f = f(\xi)$ in (67) acting via the Dirichlet boundary conditions of (62) that ensures the required state performance (66) under the fulfillments of all the constraints in (69) for *every feasible perturbation* from (65) providing a *near-optimal response* of the closed-loop control system in the case of *worst perturbations*.

The feedback control design constructed in this way leads us to the *highly nonlinear* closed-loop system (62) and (67), where $f(\xi)$ is a *discontinuous* three-positional regulator. The system may loose *robust stability* (in the large) and maintain the state performance (66) in an unacceptable *self-vibrating regime*. Developing a *variational approach* to robust stability that reduces the stability issue to a certain open-loop optimal control problem on the *infinite horizon*, we establish efficient conditions for robust stability of the closed-loop system whenever $t \geq 0$ in terms of the initial data of problem (69) and parameters of the three-positional feedback regulator. All the details can be found in [41, 42, 44].

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