Using Self-adjoint Extensions in Shape Optimization

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Abstract Self-adjoint extensions of elliptic operators are used to model the solution of a partial differential equation defined in a singularly perturbed domain. The asymptotic expansion of the solution of a Laplacian with respect to a small parameter ε is first performed in a domain perturbed by the creation of a small hole. The resulting singular perturbation is approximated by choosing an appropriate self-adjoint extension of the Laplacian, according to the previous asymptotic analysis. The sensitivity with respect to the position of the center of the small hole is then studied for a class of functionals depending on the domain. A numerical application for solving an inverse problem is presented. Error estimates are provided and a link to the notion of topological derivative is established.

1 Introduction

The standard approach in shape optimization consists in performing smooth perturbations of the boundary of a domain Ω in the normal direction. This technique does not allow topological changes in the domain. From a numerical point of view, topological changes can be obtained using levelset methods with this technique, but these changes are restricted and have no theoretical background.

In order to overcome this difficulty, several techniques have been introduced. We refer to [2] for recent developments, and to [1] for the method of homogenization in topology optimization. Other techniques rely on the simplified framework of the asymptotic analysis of the problems. In particular, the internal topology variations

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are introduced in [16], the necessary optimality conditions for simultaneous topology and shape optimization are derived in [17], the Steklov-Poincaré operators for modeling of small holes are used in [4].

The use of self-adjoint extensions of elliptic operators for modeling the solution in singularly perturbed domains was introduced in [7, 10, 11], and an alternative approach for simultaneous topology and shape optimization using self-adjoint extensions was presented in [12, 13].

In this paper we develop a numerical method based on the application of self-adjoint extensions of elliptic operators in shape optimization. The singular perturbation of the geometrical domain Ω in $\mathbb{R}^2$ is defined by a small opening ω_ε^h of diameter $O(\varepsilon)$ and of center h. The main idea of self-adjoint extensions is to model such a small defect ω_ε^h by a concentrated *action*, the so-called potential of zero-radii. In this way the solution $u_h(x,\varepsilon)$ which has a singular behavior as $\varepsilon \to 0$ is replaced by a function with the singularity at the center h of the defect. Such an approach is well-known in modeling of physical processes in material with defects, we refer the reader e.g. to [3, 8, 15]. The interesting feature of using self-adjoint extensions is that it can be extended to spectral problems and evolution boundary value problems. Also, from a numerical point of view, the singularity created by the small ε-perturbation is costly because the mesh has to be refined in the neighborhood of this small hole and the geometry of the hole has to be parameterized. To circumvent this problem, we use the concept of self-adjoint extensions of elliptic operators to define an approximation of u_h which is defined on the fixed domain Ω.

In the first chapter, the asymptotic analysis of the singularly perturbed Dirichlet problem is performed using the method of compound asymptotic expansions. The solution $u_h(x,\varepsilon)$ of the perturbed problem is approximated by a sequence of limit problems. The 2-dimensional case considered in this paper leads to a specific asymptotic analysis, due to the nature of the fundamental solution in 2D, which is of logarithm type.

In the second chapter, the self-adjoint extension of the Laplace operator with Dirichlet boundary conditions is introduced. The approximation of the solution $u_h(x,\varepsilon)$ is then the solution $\mathbf{v}_h$ of a differential equation involving the self-adjoint extension. Error estimates for this approximation with respect to ε are given. Then, for the numerical application, the sensitivity with respect to the position h of the hole is studied, and the continuity with respect to h is proven for certain functionals in L_p spaces.

In the third chapter, the numerical problem is considered. We want to minimize the L_2-distance between the approximation $\mathbf{v}_h$ and a data z measured on a subset Ω_2 of the domain Ω. The first-order derivative with respect to h of the functional is computed for use in the conjugate gradient method used in the numerical algorithm.

In the fourth chapter, a link is established with the so-called topological derivative. The topological derivative can be recovered using the self-adjoint extension model. Usually, the topological derivative can be used to solve the problem under consideration, however we show here that our algorithm is more precise than the topological derivative because it involves additional terms of the expansion of the perturbed solution. Actually, in numerical tests, our algorithm always converges to the true solution as the space step goes to zero, while the topological derivative can be quite far from the true solution.

In the fifth chapter, the numerical algorithm is presented. We use a Fletcher-Reeves conjugate gradient algorithm associated with a line search to minimize the functional. Finite differences are used to discretize the problems. Finally, in the sixth chapter, numerical results are presented.

2 Problem Formulation

Let Ω and ω, with $0 \in \omega$, $0 \in \Omega$ be two open subsets of $\mathbb{R}^2$ with smooth boundaries. Let $\varepsilon > 0$ be a small parameter and $h \in \mathbb{R}^2$. We define the perturbed domains Ω_ε^h and ω_ε^h in the following way: $\omega_\varepsilon = \{x \in \mathbb{R}^2,\ x = \varepsilon\xi,\ \xi \in \omega\}$ and $\Omega_\varepsilon = \Omega \setminus \omega_\varepsilon$, $\omega_\varepsilon^h = \{x = y + h,\ y \in \omega_\varepsilon\}$ and $\Omega_\varepsilon^h = \Omega \setminus \omega_\varepsilon^h$. We consider the following perturbed problem in $\mathbb{R}^2$, with f in $L^2(\Omega)$:

$$-\Delta u_h(x,\varepsilon) = f(x) \quad \text{in } \Omega_\varepsilon^h, \tag{1}$$

$$u_h(x,\varepsilon) = 0 \quad \text{on } \partial\Omega, \tag{2}$$

$$u_h(x,\varepsilon) = 0 \quad \text{on } \partial\omega_\varepsilon^h. \tag{3}$$

In order to approximate the solution of (1)-(3), we use the technique of *compound asymptotic expansions*. The main idea of this technique is to look for an approximation in the form of a series with respect to the power of ε, with the coefficients given by a sequence of limit problems defined either in the unperturbed domain Ω or in $\mathbb{R}^2 \setminus \omega$. The limit problems defined on $\mathbb{R}^2 \setminus \omega$ are called *boundary layers* because they correspond to solutions concentrated on the boundary of ω_ε^h and vanishing at finite distance of ω_ε^h. The boundary conditions verified by a problem are determined by the discrepancy left by the higher-order limit problem. Due to the nature of the fundamental solution in dimension 2, i.e. a logarithm, a specific procedure needs to be used, which leads to an expansion containing powers of $\ln\varepsilon$. Even if the full expansion can be obtained in the case of the Dirichlet equation we are looking at, we restrict ourselves to the first term of the expansion, which is the only term of interest for our purposes.

2.1 First Limit Problem

The first approximation v^0 solves:

$$-\Delta v^0(x) = f(x) \quad \text{in } \Omega, \tag{4}$$

$$v^0(x) = 0 \quad \text{on } \partial\Omega. \tag{5}$$

Since f is in $L^2(\Omega)$ and Ω is smooth, v^0 is in $H^2(\Omega)$. This approximation is satisfying outside a neighborhood of the boundary of the hole ω_ε^h. Due to the Dirichlet conditions on the boundary of the hole ω_ε^h, $u_h(x,\varepsilon)$ will be better approximated by

$$-\Delta v_h(x) = f(x) + \beta_h \delta(x-h) \quad \text{in } \Omega. \tag{6}$$
$$v_h(x) = 0 \quad \text{on } \partial\Omega. \tag{7}$$

We then have

$$v_h(x) = v^0(x) + \beta_h G(x,h),$$

where $G(x,y)$ is the generalized Green function defined by

$$-\Delta_x G(x,y) = \delta(x-y) \quad \text{in } \Omega, \tag{8}$$
$$G(x,y) = 0 \quad \text{on } \partial\Omega, \tag{9}$$

and $\delta(x-y)$ is the Dirac mass at y. The function $u_h(x,\varepsilon)$ is then approximated outside a neighborhood of ω_ε^h by

$$u_h(x,\varepsilon) \backsimeq v^0(x) + \beta_h G(x,h).$$

The function G admits the following representation:

$$G(x,h) = -\left\{(2\pi)^{-1}\log|x-h| + \mathscr{G}(x,h)\right\}, \tag{10}$$

where $|\cdot|$ stands for the euclidean norm in $\mathbb{R}^2$. The function $\mathscr{G}$ is the regular part of the Green function solution of

$$-\Delta_x \mathscr{G}(x,y) = 0 \quad \text{in } \Omega, \tag{11}$$
$$\mathscr{G}(x,y) = -(2\pi)^{-1}\log|x-y| \quad \text{on } \partial\Omega, \tag{12}$$

For $x \in \partial\Omega$ and as $h \to 0$, we can use Taylor's formula to expand $-(2\pi)^{-1}\log|x-h|$ in (12) with respect to h and obtain

$$-(2\pi)^{-1}\log|x-h| = -(2\pi)^{-1}\log|x| + (2\pi)^{-1}\langle h, \frac{x}{|x|^2}\rangle + r_h,$$

with $\|r_h\|_{L^\infty(\partial\Omega)} = O(|h|^2)$. Thus $\mathscr{G}(x,h)$ admits the expansion

$$\mathscr{G}(x,h) = \mathscr{G}(x,0) + \mathscr{S}_h(x) + \mathscr{R}_h(x), \tag{13}$$

with $\mathscr{S}_h(x)$ and $\mathscr{R}_h(x)$ solutions of

$$-\Delta \mathscr{S}_h(x) = 0 \quad \text{in } \Omega, \tag{14}$$
$$\mathscr{S}_h(x) = (2\pi)^{-1}\langle h, \frac{x}{|x|^2}\rangle \quad \text{on } \partial\Omega. \tag{15}$$

$$-\Delta\mathscr{R}_h(x) = 0 \quad \text{in } \Omega, \tag{16}$$
$$\mathscr{R}_h(x) = r_h \quad \text{on } \partial\Omega. \tag{17}$$

Finally, $\mathscr{G}(h,h)$ can be decomposed into

$$\mathscr{G}(h,h) = \mathscr{G}(0,0) + \langle h, \nabla\mathscr{G}(0,0)\rangle + \mathscr{S}_h(0) + \mathscr{R}_h(h) + O(|h|^2). \tag{18}$$

Since $\|r_h\|_{L^\infty(\partial\Omega)} = O(|h|^2)$ we get $\|\mathscr{R}_h\|_{L^\infty(\Omega)} = O(|h|^2)$. We also have $\|\mathscr{S}_h\|_{L^\infty(\Omega)} = O(|h|)$. The approximation $u_h(x,\varepsilon) \simeq v^0(x) + \beta_h G(x,h)$ does not verify the boundary condition (3) on the hole. Consequently, a boundary layer $w_h^0(\xi_h,\varepsilon)$ must be added, which depends on the fast variable ξ_h defined as $\xi_h = \varepsilon^{-1}(x-h)$, in order to compensate for the induced discrepancy. Expanding $v^0(x) + \beta_h G(x,h)$ when $x \to h$ we get

$$\begin{aligned} v^0(x) + \beta_h G(x,h) &= v^0(x) - \beta_h \left\{(2\pi)^{-1}\log|x-h| + \mathscr{G}(x,h)\right\} \\ &= v^0(h) - \beta_h \left\{(2\pi)^{-1}\log|\varepsilon\xi_h| + \mathscr{G}(h,h)\right\} + z_\varepsilon^h(x). \end{aligned} \tag{19}$$

The estimates on the rest $z_\varepsilon^h(x)$ will be addressed later in Section 3.2. In view of (19), we introduce the boundary layer $w_h^0(\xi,\varepsilon)$ solution of the following system

$$-\Delta_{\xi_h} w_h^0(\xi_h,\varepsilon) = 0 \quad \text{in } \mathbb{R}^2 \setminus \omega_h, \tag{20}$$
$$w_h^0(\xi_h,\varepsilon) = -v^0(h) + \beta_h \left\{(2\pi)^{-1}\log|\varepsilon\xi_h| + \mathscr{G}(h,h)\right\} \quad \text{on } \partial\omega_h. \tag{21}$$

The solution of (20)-(21) is

$$w_h^0(\xi_h,\varepsilon) = -v^0(h) + \beta_h \left\{(2\pi)^{-1}\log\varepsilon + \mathscr{G}(h,h)\right\} + \beta_h \mathscr{E}_h^0(\xi_h), \tag{22}$$

with $\mathscr{E}_h^0(\xi_h)$ solution of

$$-\Delta_{\xi_h} \mathscr{E}_h^0(\xi_h) = 0 \quad \text{in } \mathbb{R}^2 \setminus \omega_h, \tag{23}$$
$$\mathscr{E}_h^0(\xi_h) = (2\pi)^{-1}\log|\xi_h| \quad \text{on } \partial\omega_h. \tag{24}$$

The function $\mathscr{E}_h^0(\xi_h)$ admits the following expansion w.r.t. ξ_h

$$\mathscr{E}_h^0(\xi_h) = (2\pi)^{-1}L + O(|\xi_h|^{-1}), \tag{25}$$

where L is a constant depending only on the shape of ω. The quantity $\exp(L)$ is called the logarithmic capacity of ω. Thus, $w_h^0(\xi_h,\varepsilon)$ admits the expansion

$$w_h^0(\xi_h,\varepsilon) = -v^0(h) + \beta_h \left\{(2\pi)^{-1}\log\varepsilon + \mathscr{G}(h,h)\right\} + \beta_h(2\pi)^{-1}L + O(|\xi_h|^{-1}).$$

In order to have $w_h^0(\xi_h,\varepsilon) \to 0$ as $|\xi_h| \to \infty$, a condition is imposed on β_h:

$$\beta_h = \left\{(2\pi)^{-1}(\log\varepsilon + L) + \mathscr{G}(h,h)\right\}^{-1} v^0(h), \tag{26}$$

so that

$$w_h^0(\xi_h,\varepsilon)=\beta_h\mathscr{E}_h^0(\xi_h).$$

As a consequence, the solution $u_h(x,\varepsilon)$ of (1)-(3) can be represented by

$$u_h(x,\varepsilon)=v^0(x)+\beta_h G(x,h)+\tilde{u}_h^0(x,\varepsilon), \tag{27}$$

where the function $\tilde{u}_h^0(x,\varepsilon)$ is solution of the following problem

$$-\Delta\tilde{u}_h^0(x,\varepsilon)=0 \quad \text{in } \Omega_\varepsilon^h \tag{28}$$

$$\tilde{u}_h^0(x,\varepsilon)=0 \quad \text{on } \partial\Omega \tag{29}$$

$$\begin{aligned}\tilde{u}_h^0(x,\varepsilon)=&-\left(v^0(x)-v^0(h)\right)\\ &+\beta_h\left\{(2\pi)^{-1}(\log|\xi_h|-L)\right\}\\ &+\beta_h\left\{\mathscr{G}(x,h)-\mathscr{G}(h,h)\right\} \quad \text{on } \partial\omega_\varepsilon^h.\end{aligned} \tag{30}$$

3 Self-adjoint Extension of the Laplacian with Dirichlet Conditions

3.1 Self-adjoint Extension

For the sake of simplicity, we assume that $h=0$ in what follows (without loss of generality) and we will return to the general case in the next section. In what follows, we use the notation β instead of β_0. The self-adjoint extension of the Laplace operator with Dirichlet boundary conditions is defined as follows: let $\mathscr{A}_0$ be the Laplacian operator $-\Delta_x$ in $L_2(\Omega)$ with the domain of definition

$$\mathscr{D}(\mathscr{A}_0)=\left\{v\in C_0^\infty(\overline{\Omega}\setminus\{0\})\right\} \tag{31}$$

The inclusion $v\in\mathscr{D}(\mathscr{A}_0)$ indicates that v satisfies the boundary conditions (3) and is equal to zero in the neighborhood of the center 0 of ω_ε, this last condition mimicking the Dirichlet condition (3).

Introduce the cut-off function $\chi_\delta(x)=\chi(\delta x)$ where χ is such that $\chi\in C^\infty(\mathbb{R}^2)$ and

$$\chi(x)=1 \text{ for } |x|<1, \tag{32}$$

$$\chi(x)=0 \text{ for } |x|>2. \tag{33}$$

We assume that δ is chosen such that χ_δ has compact support in Ω. The closure $\overline{\mathscr{A}_0}$ and the adjoint $\mathscr{A}_0^*$ of the operator $\mathscr{A}_0$ are given by the following lemma:

Lemma 1. *The closure $\overline{\mathscr{A}_0}$ and the adjoint $\mathscr{A}_0^*$ of the operator $\mathscr{A}_0$ are given by the differential expression $-\Delta_x$, with the respective domain of definition:*

$$\mathscr{D}(\overline{\mathscr{A}_0}) = \left\{ v \in H^2(\Omega),\ v(0) = 0,\ v = 0 \text{ on } \partial\Omega \right\} \tag{34}$$

and

$$\mathscr{D}(\mathscr{A}_0^*) = \left\{ v\ :\ v(x) = \chi_\delta(x)\left(-\frac{a}{2\pi}\log r + b\right) + \bar{v}(x),\ \bar{v} \in \mathscr{D}(\overline{\mathscr{A}_0}),\ a,b \in \mathbb{R} \right\} \tag{35}$$

Note that in (35), it can be shown that the domain $\mathscr{D}(\mathscr{A}_0^*)$ does not depend on the cut-off function χ_δ. Since the domain of definition of the initial operator $\mathscr{A}_0$ is restricted, the domain of definition of the adjoint is large, and the two operators $\overline{\mathscr{A}_0}$ and $\mathscr{A}_0^*$ are not self-adjoints. However, there exists a family of self-adjoint operators $\mathscr{A}$, such that $\mathscr{A}_0 \subset \mathscr{A} \subset \mathscr{A}_0^*$ and the domain of definition $\mathscr{D}(\mathscr{A})$ contains all the required singular solutions for the Dirichlet problem in Ω.

The family of self-adjoint extensions of the operator $\mathscr{A}_0$ is built by restricting the domain of the operator $\mathscr{A}_0^*$. The abstract boundary condition $b = Sa$ is added in the definition of $\mathscr{D}(\mathscr{A}_0^*)$ with a given coefficient S. In our case, we will obtain S depending on the asymptotic expansion of v_h w.r.t. ε. With such an S, the influence of the small hole can be modeled. Therefore, the following theorem can be proved.

Theorem 1. *Let* $\mathbf{A}$ *be the restriction of the operator* $\mathscr{A}_0^*$ *to the vector space*

$$\mathscr{D}(\mathbf{A}) = \{ v \in \mathscr{D}(\mathscr{A}_0^*)\ :\ b = Sa \} \tag{36}$$

where $S = S(\varepsilon) = (2\pi)^{-1}(\log\varepsilon + L)$, *$L$ is a constant which depends on the shape of* ω*. Then* $\mathbf{A}$ *is a self-adjoint operator.*

The following equation

$$\mathbf{A}\mathbf{v} = f \in L_2(\Omega) \tag{37}$$

admits a unique solution $\mathbf{v} \in \mathscr{D}(\mathbf{A})$ *and* $\mathbf{v}$ *is given by*

$$\mathbf{v}(x) = v^0(x) + \beta G(x,0) \qquad \forall x \in \Omega.$$

Proof. 1) It is enough to prove the following: if for $\mathbf{v}, f \in L_2(\Omega)$ the following equality is true

$$(\mathbf{v}, \mathbf{A}\mathbf{z})_\Omega = (f, \mathbf{z})_\Omega \qquad \forall \mathbf{z} \in \mathscr{D}(\mathbf{A}), \tag{38}$$

then $\mathbf{v} \in \mathscr{D}(\mathbf{A})$ and $f = \mathbf{A}\mathbf{v}$. Since $\mathscr{A}_0 \subset \mathbf{A}$, we can see that $\mathbf{v} \in \mathscr{D}(\mathscr{A}_0^*)$ and $\mathscr{A}_0^*\mathbf{v} = f$. Thus, it is only necessary to show that $\mathbf{v} \in \mathscr{D}(\mathbf{A})$. In view of (38) we can write the Green's formula:

$$0 = (\mathbf{v}, \mathbf{A}\mathbf{z})_\Omega - (\mathscr{A}_0^*\mathbf{v}, \mathbf{z})_\Omega \tag{39}$$

$$= \lim_{\delta\to 0} \int_{\Omega\setminus\mathbb{B}_\delta} (\mathbf{z}\Delta_x\mathbf{v} - \mathbf{v}\Delta_x\mathbf{z})\, dx \tag{40}$$

$$= \lim_{\delta\to 0} \int_{\partial\mathbb{B}_\delta} \mathbf{v}\partial_n\mathbf{z} - \mathbf{z}\partial_n\mathbf{v}\, ds_x + \int_{\partial\Omega} \mathbf{v}\partial_n\mathbf{z} - \mathbf{z}\partial_n\mathbf{v}\, ds_x \tag{41}$$

$$= \lim_{\delta\to 0} \int_{\partial\mathbb{B}_\delta} \mathbf{v}\partial_n\mathbf{z} - \mathbf{z}\partial_n\mathbf{v}\, ds_x. \tag{42}$$

Since $\mathbf{v} \in \mathscr{D}(\mathscr{A}_0^*)$, $v = 0$ on $\partial\Omega$ and as a consequence $\int_{\partial\Omega} \mathbf{v}\partial_n \mathbf{z} - \mathbf{z}\partial_n \mathbf{v}\, ds_x = 0$. In what follows we introduce the notation $r = |x|$. Replacing $\mathbf{v}$ and $\mathbf{z}$ by the asymptotic expansions given in the definition of $\mathscr{D}(\mathscr{A}_0^*)$, with the coefficients denoted respectively a, b and p, q, we get

$$
\begin{aligned}
0 = & \lim_{\delta\to 0} \delta \int_0^{2\pi} (b - a\frac{1}{2\pi}\log r)\frac{\partial}{\partial r}(q - p\frac{1}{2\pi}\log r)_{|_{r=\delta}} \\
& - (q - p\frac{1}{2\pi}\log r)\frac{\partial}{\partial r}(b - a\frac{1}{2\pi}\log r)_{|_{r=\delta}}\, d\phi \\
= & \lim_{\delta\to 0} a(q - p\frac{1}{2\pi}\log\delta) - p(b - a\frac{1}{2\pi}\log\delta) \\
= & \, aq - bp \\
= & \, (Sa - b)p,
\end{aligned} \tag{43}
$$

and the conclusion is $b = Sa$ which means that $\mathbf{v} \in \mathscr{D}(\mathbf{A})$. Here we have used the relation $q = Sp$ since $\mathbf{z} \in \mathscr{D}(\mathbf{A})$.
2) First of all, the unicity of the solution is proved. Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be two functions in $\mathscr{D}(\mathbf{A})$. Then the difference $\mathbf{v} = \mathbf{v_1} - \mathbf{v_2}$ verifies $\mathbf{Av} = 0$ in Ω and $\mathbf{v} = 0$ on $\partial\Omega$. Thus $\mathbf{v}$ is the fundamental solution of the Laplacian

$$
\mathbf{v} = \mu G(x,0) = -\mu\left((2\pi)^{-1}\log r + \mathscr{G}(x,0)\right) \tag{44}
$$

where G and $\mathscr{G}$ are defined in (8)-(9) and (11)-(12). The asymptotic representation (44) gives coefficients $a = \mu$ and $b = -\mathscr{G}(0,0)\mu$. From the definition of $\mathscr{D}(\mathbf{A})$ we get $b = Sa$, thus we obtain

$$
-\mathscr{G}(0,0)\mu = ((2\pi)^{-1}\log\varepsilon + L)\mu
$$

which implies $\mu = 0$. Thus $\mathbf{v} \equiv 0$ and we have proved unicity of the solution.

Now it remains to show that

$$
\mathbf{v}(x) = v^0(x) + \beta G(x,0) \tag{45}
$$

is solution of $\mathbf{Av} = f$. In view of definitions (8)-(9) and (4)-(5) of $G(x,0)$ and v^0, respectively, we clearly have $-\Delta_x u(x,\varepsilon) = f(x)$ in $\Omega\setminus 0$ and $u(x,\varepsilon) = 0$ on $\partial\Omega$. We also have $a = \beta$ and $b = v^0(0) - \beta\mathscr{G}(0,0)$, so that the relation $b = Sa$ is satisfied. As a consequence, we get $\mathbf{v} \in \mathscr{D}(\mathbf{A})$. □

3.2 Estimates

From now on, we will write $\mathbf{v}_h$ instead of $\mathbf{v}$ to stress the dependence of $\mathbf{v}$ on h. According to (37), $\mathbf{v}_h$ corresponds to the first-order terms in the expansion (27). Therefore we will now give an estimate for the L^2-norm of $\tilde{u}_h^0 = u_h - \mathbf{v}_h$. Define $\tilde{u}_h^1$, $\tilde{u}_h^2$ and $\tilde{u}_h^3$ harmonic functions on Ω_ε such that

$$\tilde{u}_h^0 = \tilde{u}_h^1 + \tilde{u}_h^2 + \tilde{u}_h^3$$

and for all $x \in \partial \omega_\varepsilon^h$ we have in view of (28)-(30)

$$\tilde{u}_h^1(x,\varepsilon) = -(v^0(x) - v^0(h)), \tag{46}$$

$$\tilde{u}_h^2(x,\varepsilon) = \beta_h \left\{ (2\pi)^{-1}(\log|\xi_h| - L) \right\}, \tag{47}$$

$$\tilde{u}_h^3(x,\varepsilon) = \beta_h \left\{ \mathscr{G}(x,h) - \mathscr{G}(h,h) \right\}. \tag{48}$$

Since $f \in L^2(\Omega)$, $v^0 \in H^2(\Omega)$ and by the Sobolev-Rellich theorem, we have

$$v^0 \in C^0(\overline{\Omega}) \text{ and } \nabla v^0 \in L^\infty(\Omega).$$

Therefore we get

$$\sup_{x \in \partial \omega_\varepsilon^h} |v^0(x) - v^0(h)| \le M_1 \varepsilon \sup_{x \in \omega_\varepsilon^h} |\nabla v^0(x)|,$$

where M_1 depends only on the shape of ω; $M_1 = 1$ if $\omega = B(0,1)$. By the maximum principle we get

$$\|\tilde{u}_h^1\|_{L^\infty(\Omega_\varepsilon)} \le M_1 \varepsilon \sup_{x \in \omega_\varepsilon^h} |\nabla v^0(x)|$$

and

$$\|\tilde{u}_h^1\|_{L^2(\Omega_\varepsilon)} \le M_1 \varepsilon \sup_{x \in \omega_\varepsilon^h} |\nabla v^0(x)|,$$

with M_1 depending only on the shape of Ω and ω. In a similar way, since $\mathscr{G}(\cdot,h) \in C^\infty(\overline{\Omega})$, we have

$$\sup_{x \in \partial \omega_\varepsilon^h} |\mathscr{G}(x,h) - \mathscr{G}(h,h)| \le M_3 \varepsilon \sup_{x \in \omega_\varepsilon^h} |\nabla \mathscr{G}(x,h)|,$$

and

$$\|\tilde{u}_h^3\|_{L^\infty(\Omega_\varepsilon)} \le M_3 \beta_h \varepsilon \sup_{x \in \omega_\varepsilon^h} |\nabla \mathscr{G}(x,h)|,$$

$$\|\tilde{u}_h^3\|_{L^2(\Omega_\varepsilon)} \le M_3 \beta_h \varepsilon \sup_{x \in \omega_\varepsilon^h} |\nabla \mathscr{G}(x,h)|,$$

with M_3 depending only on the shape of Ω and ω. Now consider the case of $\tilde{u}_h^2$. If $\omega = B(0,1)$, we get $\tilde{u}_h^2 \equiv 0$. In the more general case of any shape for ω, we get according to (25)

$$\tilde{u}_h^2 = O(|\xi_h|^{-1}),$$

and since

$$\int_{\Omega_\varepsilon^h} |\xi_h|^{-2} dx = \int_{\Omega_\varepsilon^h} \varepsilon^2 |x-h|^{-2} dx \le \tilde{M}_2 \varepsilon^2 |\log \varepsilon|,$$

where $\tilde{M}_2$ is some constant independent of ε and h. In view of the expression of β_h we get

$$\|\tilde{u}_h^2\|_{L^2(\Omega_\varepsilon)} \leq \tilde{M}_2|\beta_h|\varepsilon|\log\varepsilon|^{\frac{1}{2}} \leq M_2|v^0(h)|\varepsilon,$$

and M_2 depends only on the shape of Ω and ω. Gathering the estimates for $\tilde{u}_h^1, \tilde{u}_h^2$ and $\tilde{u}_h^3$, we obtain

$$\|\tilde{u}_h^0\|_{L^2(\Omega_\varepsilon)} \leq M\varepsilon, \tag{49}$$

with M depending only on the shape of Ω and ω.

3.3 Derivative with Respect to the Position of the Hole

Recall that the function $u_h(x,\varepsilon)$ of (1)-(3) can be represented by

$$u_h(x,\varepsilon) = v^0(x) + \beta_h G(x,h) + \tilde{u}_h^0(x,\varepsilon). \tag{50}$$

It can also be represented in a form derived from (35)

$$u_h(x,\varepsilon) = -\frac{a_h}{2\pi}\log r_h + b_h + \bar{u}_h(x,\varepsilon) \tag{51}$$

where

$$a_h = \beta_h = \left\{(2\pi)^{-1}(\log\varepsilon + L) + \mathscr{G}(h,h)\right\}^{-1} v^0(h), \tag{52}$$

$$b_h = v^0(h) - \beta_h\mathscr{G}(h,h) = Sa_h \tag{53}$$

with $S = (2\pi)^{-1}(\log\varepsilon + L)$, and (r_h,θ_h) stand for the polar coordinates of center h. The coefficient S depends on ε but does not depend on the position of the hole h. The function $\bar{u}_h$ belongs to the set $\mathscr{D}(\overline{\mathscr{A}_0})$.

3.4 Energy Functionals in L_p

We consider functionals of the form:

$$\mathscr{F}(u,\varepsilon) = \int_{\Omega_\varepsilon} F(x,u)\,dx \tag{54}$$

with $u \in \mathscr{D}(\mathbf{A})$. We make an assumption on the functional (54), sufficient for further asymptotic analysis. Namely, the following inequality holds for some $p \in [1,\infty[$ and for all $u,v \in L_p(\Omega_\varepsilon)$

$$|\mathscr{F}(u,\varepsilon) - \mathscr{F}(v,\varepsilon)| \leq c\|u-v\|_{L_p(\Omega_\varepsilon)}\left(\|u\|^{p-1}_{L_p(\Omega_\varepsilon)} + \|v\|^{p-1}_{L_p(\Omega_\varepsilon)}\right), \tag{55}$$

where the constant c depends on Ω, but it is independent of the parameter ε and of the functions u,v. We assume also that the same inequality holds in unperturbed domain Ω,

$$|\mathscr{F}(u,0)-\mathscr{F}(v,0)|\leq c\|u-v\|_p\left(\|u\|_p^{p-1}+\|v\|_p^{p-1}\right), \tag{56}$$

where $\|\ .\ \|_p$ denotes the norm in $L_p(\Omega)$. Let $u_h(x,\varepsilon)$ be the solution of equation (1)-(3). The function u_h is extended by zero over the opening ω_ε^h, and the extended function is still denoted u_h. Since $u_h \in H_0^1(\Omega_\varepsilon)$ we also have $u_h \in H_0^1(\Omega)$; see [6, Prop. 3.1.4, p. 78]. Then, thanks to the imbedding $H^1(\Omega)\subset L_p(\Omega),\ p\in[1,\infty[$ and to inequality (56) we have

$$|\mathscr{F}(u,0)-\mathscr{F}(u_h,0)|\leq c\|u-u_h\|_p\left(\|u\|_p^{p-1}+\|u_h\|_p^{p-1}\right). \tag{57}$$

According to (51) we can write

$$\|u-u_h\|_p=\left\|-\frac{a_0}{2\pi}\log r+\frac{a_0}{2\pi}\log r_h+S(a_0-a_h)+\bar{u}-\bar{u}_h\right\|_p. \tag{58}$$

In view of the expansion (18) of $\mathscr{G}(h,h)$ and the smoothness of v^0 solution of (4)-(5) we get

$$\|a_0-a_h\|_p=\|\beta-\beta_h\|_p\leq c|h| \tag{59}$$

where c is a constant depending only on Ω, for ε small enough, according to the expression (52) of β_h. For b_h we obtain a similar result because of the relation $b_h = Sa_h$

$$\|b_0-b_h\|_p=\|Sa-Sa_h\|_p\leq Sc|h| \tag{60}$$

and $S=(2\pi)^{-1}(\log\varepsilon+L)$. Since $\bar{u}$ and $\bar{u}_h$ belong to $\mathscr{D}(\overline{\mathscr{A}_0})$, $\bar{u}$ and $\bar{u}_h$ are in $H^2(\Omega)$, and we obtain the same inequality for $\|\bar{u}-\bar{u}_h\|_p$

$$\|\bar{u}-\bar{u}_h\|_p\leq c|h|. \tag{61}$$

The only term that remains to estimate in (58) is $-\frac{a_0}{2\pi}\log r+\frac{a_0}{2\pi}\log r_h$, since we have proven the continuity of $\|a_0-a_h\|_p$ in (59), we only have to estimate $\|\log r-\log r_h\|_p$. Possibly changing the coordinates, we may suppose that $h=(|h|,0)$. Then we can split the following integral into two parts

$$\int_\Omega |\log r_h-\log r|^p\,dx = I_0^h+I_1^h$$

with

$$\begin{aligned} I_0^h &= \int_{|x|<2|h|} |\log|x+|h|e_1|-\log|x||^p\,dx \\ &= |h|^2\int_{|\xi|<2} |\log|\xi+e_1|-\log|\xi||^p\,d\xi \\ &\leq c|h|^2. \end{aligned} \tag{62}$$

We have used the change of variables $x=|h|\xi$. We also have $e_1=(1,0)$. Further,

$$\begin{aligned} I_1^h &= \int_{\Omega\setminus|x|<2|h|} \left|\log\frac{|x|^2-2|h|x_1+|h|^2}{|x|^2}\right|^p dx \\ &\le C_\alpha \int_{\Omega\setminus|x|<2|h|} \left(\left(\frac{|h|}{r}\right)^{\alpha p} + \left(\frac{|h|^2}{r^2}\right)^{\alpha p}\right) \\ &\le C_\alpha \left(|h|^{\alpha p}\int_{2|h|}^{D} r^{-\alpha p+1}\,dr + |h|^{2\alpha p}\int_{2|h|}^{D} r^{-2\alpha p+1}\,dr\right), \end{aligned} \tag{63}$$

with $\alpha \in]0,1]$ and C_α is a constant depending only on α. Thus, if $\alpha p < 1$, then

$$I_1^h \le C_\alpha \left(|h|^{\alpha p}|h|^{-\alpha p+2} + |h|^{2\alpha p}|h|^{-2\alpha p+2}\right) = C_\alpha |h|^2.$$

Finally we obtain

$$\|\log r - \log r_h\|_p \le C_\alpha |h|^{2/p}. \tag{64}$$

Therefore, choosing $p \in]1,2]$ and combining (59), (60), (61) and (64) we obtain

$$|\mathscr{F}(u,0) - \mathscr{F}(u_h,0)| \le M|h|, \tag{65}$$

where M is a constant which depends only on the shape of Ω.

4 Least Squares Functional

In this section, the domain Ω is split into two disjoint open sets Ω_1 and Ω_2 so that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, and we introduce a least squares functional $\mathscr{J}_\varepsilon$ which measures the L_2 distance between some data z and an approximation $\mathbf{v}_h$ of (1)-(3) on Ω_2 (see Fig. 4). The data z corresponds to the solution in a domain with a hole ω_ε^h whose position in Ω_1 is unknown. In what follows, we assume that ε is known. By minimizing $\mathscr{J}_\varepsilon$ w.r.t. h, we are able to find the position of the hole ω_ε^h.

From now on, the domain ω is assumed to be a ball of radius 1. The general case is easily deduced from this particular case. We consider a cost functional defined as follows:

$$\mathscr{J}_\varepsilon(h) := \frac{1}{2}\int_{\Omega_2} (\mathbf{v}_h(x) - z(x))^2 dx \tag{66}$$

where z is a given observation in $L^2(\Omega_2)$ and $\mathbf{v}_h$ is given by:

$$\mathbf{v}_h(x) = v^0(x) + \frac{G(x,h)}{\frac{\log\varepsilon + L}{2\pi} + \mathscr{G}(h,h)} v^0(h), \quad \forall x \in \Omega. \tag{67}$$

In the case where ω_ε is the ball $B(x_0,\varepsilon)$, we get $|\xi_h| = 1$ on $\partial\omega_h$ and therefore $L = 0$. We want to solve the minimization problem

$$\min_{h\in\Omega_1} \mathscr{J}_\varepsilon(h). \tag{68}$$

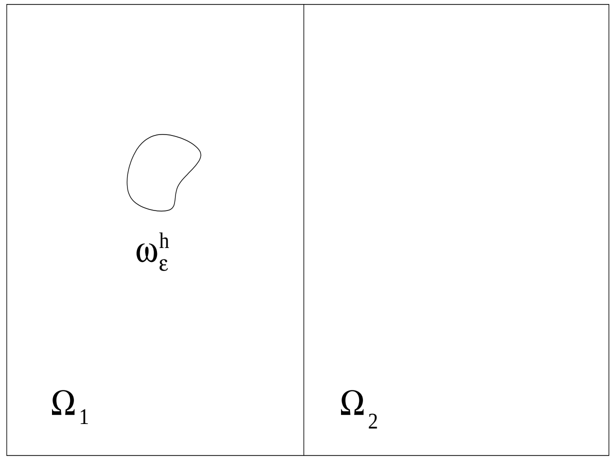

Fig. 1 The domain ω

To this end we use a Fletcher-Reeves algorithm with a line search procedure, therefore we must first compute the gradient of $\mathscr{J}_\varepsilon$ w.r.t. h.

$$\nabla \mathscr{J}_\varepsilon(h) = \int_{\Omega_2} (\mathbf{v}_h(x) - z(x)) \nabla_h \mathbf{v}_h(x) dx \tag{69}$$

where ∇_h denotes the gradient with respect to h. To compute $\nabla_h \mathbf{v}_h(x)$ let us simplify (67) by introducing:

$$\lambda(h) = \left(\frac{\log \varepsilon + L}{2\pi} + \mathscr{G}(h,h) \right)^{-1}. \tag{70}$$

Thus (67) can be written as

$$\mathbf{v}_h(x) = v^0(x) + \lambda(h) G(x,h) v^0(h) \quad \forall x \in \Omega. \tag{71}$$

The gradient $\nabla_h \mathbf{v}_h(x)$ takes the form:

$$\begin{aligned} \nabla_h \mathbf{v}_h(x) = \lambda(h) &\left[v^0(h) \left(\frac{x-h}{2\pi r_h^2} - \nabla_y \mathscr{G}(x,h) \right) + G(x,h) \nabla v^0(h) \right] \\ &\qquad - \lambda(h)^2 G(x,h) v^0(h) \nabla_h [\mathscr{G}(h,h)], \\ = \lambda(h) &\left[v^0(h) \left(\frac{x-h}{2\pi r_h^2} - \nabla_y \mathscr{G}(x,h) \right) + G(x,h) \nabla v^0(h) \right] \\ &\qquad - \lambda(h)^2 G(x,h) v^0(h) [\nabla_x \mathscr{G}(h,h) + \nabla_y \mathscr{G}(h,h)]. \end{aligned} \tag{72}$$

where $r_h = |x-h|$, and $\nabla_x \mathscr{G}$ and $\nabla_y \mathscr{G}$ are the gradients with respect to the first and second variables of $\mathscr{G}$, respectively. The value of $\nabla_x \mathscr{G}$ is clearly defined according to (11) and (12), and $\nabla_y \mathscr{G}$ is solution of

$$\begin{aligned} -\Delta_x[\nabla_y \mathscr{G}](x,y) &= 0 \quad \text{in} \quad \Omega \\ \nabla_y \mathscr{G}(x,y) &= (2\pi)^{-1} \tfrac{x-y}{\|x-y\|^2} \text{ on } \partial\Omega. \end{aligned} \tag{73}$$

5 Topological Derivative

A new idea was introduced first by Schumacher in 1994 with the so-called "bubble method", where the parameterized setting is kept and holes are created in the domain according to a certain criterion. This idea was later developed by Sokołowski and Żochowski [16], and Guillaume and Masmoudi [4], with the introduction of the *topological derivative*. The topological derivative measures the variation of a cost functional depending on the shape of a domain, when a small change in the topology of this domain is performed, for instance with the creation of a small hole of any shape.

Let Ω and ω_ε be two open sets in $\mathbb{R}^N$, and $\omega_\varepsilon \subset B(h,\varepsilon)$ where $B(h,\varepsilon)$ is a ball of radius $\varepsilon > 0$ centered at $h \in \Omega$. Denote $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$, if the cost functional $J(\Omega)$ is differentiable with respect to the creation of this small hole ω_ε, then we can write the expansion

$$J(\Omega_\varepsilon) = J(\Omega) + \rho(\varepsilon)\mathscr{T}(h) + o(\rho(\varepsilon)), \tag{74}$$

with $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, $\rho(\varepsilon) > 0$, and $\mathscr{T}(h)$ is the so-called topological derivative of J. First note that the topological derivative is a pointwise expression defined at every point of the domain, therefore it is usually easy to compute and gives an efficient criteria for a descent direction in a gradient method.

There is a link between the self-adjoint extension introduced in Section 3 and the notion of topological derivative. Indeed, both are obtained through the asymptotic expansion w.r.t. ε of the solution of (1)-(3). Actually, it is possible to recover the usual topological derivative from the self-adjoint extension model. In order to do so, we write the expansion with respect to ε of the functional $\mathscr{J}_\varepsilon$:

$$\begin{aligned} \mathscr{J}_\varepsilon(h) &= \frac{1}{2}\int_{\Omega_2} (\mathbf{v}_h(x) - z(x))^2 dx \\ &= \frac{1}{2}\int_{\Omega} (\mathbf{v}_h(x) - z(x))^2 \mathbb{1}_{\Omega_2}(x) dx \\ &= \frac{1}{2}\int_{\Omega} (v^0(x) - z(x))^2 \mathbb{1}_{\Omega_2}(x) dx \\ &\quad + \int_{\Omega} (v^0(x) - z(x))(\lambda(h)G(x,h)v^0(h))\mathbb{1}_{\Omega_2}(x) dx + \mathscr{R}_1(\varepsilon,h), \end{aligned} \tag{75}$$

with

$$\mathscr{R}_1(\varepsilon,h) = (\lambda(h)v^0(h))^2 \int_{\Omega_2} G(x,h)^2 dx \leq M(\log\varepsilon)^{-2}, \tag{76}$$

and M depends only on the shape of Ω and Ω_2. Then we introduce the following adjoint state p

$$-\Delta p(x) = (v^0(x) - z(x))\mathbb{1}_{\Omega_2}(x) \quad \text{in } \Omega, \tag{77}$$
$$p(x) = 0 \quad \text{on } \partial\Omega, \tag{78}$$

and after an integration by parts we obtain

$$\mathscr{J}_\varepsilon(h) = \frac{1}{2}\int_{\Omega_2}(v^0(x) - z(x))^2 dx + \lambda(h)v^0(h)p(h) + \mathscr{R}_1(\varepsilon, h), \tag{79}$$

Since $\lambda(h)$ admits the expansion

$$\lambda(h) = 2\pi(\log\varepsilon)^{-1} + O((\log\varepsilon)^{-2}) \quad \text{as} \quad \varepsilon \to 0,$$

we can expand (79) further:

$$\mathscr{J}_\varepsilon(h) = \frac{1}{2}\int_{\Omega_2}(v^0(x) - z(x))^2 dx - 2\pi|\log\varepsilon|^{-1}v^0(h)p(h) + \mathscr{R}_2(\varepsilon, h), \tag{80}$$

with

$$\mathscr{R}_2(\varepsilon, h) = O((\log\varepsilon)^{-2}) \quad \text{as} \quad \varepsilon \to 0. \tag{81}$$

Therefore we have obtained expansion (74) with $\rho(\varepsilon) = 2\pi|\log\varepsilon|^{-1}$ and $\mathscr{T}(h) = -v^0(h)p(h)$. We will use this result later to initialize our algorithm.

Remark 1 *As expected, the topological derivative $\mathscr{T}(h)$ is easy to compute because it requires only the solution of two Laplacian equations on Ω. The drawback is that the rest $\mathscr{R}_2(\varepsilon, h)$ is of order $(\log\varepsilon)^{-2}$, which is not negligible for numerical calculations compared to the main term in $(\log\varepsilon)^{-1}$. Therefore, there is a lack of precision when using the topological derivative to localize the point, and we will see in our numerical tests that this lack of precision can lead to inaccurate results.*

6 Algorithm

The observation z in the functional $\mathscr{J}_\varepsilon$ corresponds to measured data. For the numerical tests, the position of the hole is known beforehand, and the observation is computed accordingly.

One should note that if h^* is the real position of the center of the hole, used to compute the function z, we cannot expect h^* to be the solution of the minimization problem (68) because the corresponding solution $\mathbf{v}_{h^*}$ is only an approximation of z. One can be more precise by looking at $\mathscr{J}_\varepsilon(h^*)$:

$$\mathscr{J}_\varepsilon(h^*) = \frac{1}{2}\int_{\Omega_2}(\mathbf{v}_{h^*}(x) - z(x))^2 dx.$$

According to (49), there exists M such that

$$\|v_{h^*} - z\|_{L^2(\Omega_\varepsilon)} \leq M\varepsilon,$$

and as a consequence

$$\mathscr{J}_\varepsilon(h^*) \leq M^2\varepsilon^2,$$

which means that h^* is not necessarily the optimal solution.

Usually, the topological derivative can be used to find the position of the hole. A possible way to proceed is to compute the topological derivative $\mathscr{T}(h)$ at every point $h \in \Omega_1$, and look for the minimum of $\mathscr{T}(h)$ which should give an approximate position of the unknown hole. Unfortunately, due to the lack of information since z is known only on Ω_2, and according to Remark 1, the numerical tests show that this is not enough to find the exact position of the hole. It is possible to go further in the expansion of the topological derivative to have a better approximation, but then we obtain a function which is difficult to evaluate at every point of the domain because for every h, one needs to solve a Laplace equation (to find the function $\mathscr{G}$ and compute $\mathscr{G}(h,h)$).

Therefore, the idea of the algorithm is to initialize the position of the hole by using the topological derivative, and then to use a Fletcher-Reeves conjugate gradient. This can be related to the speed method in shape optimization, since once the hole is created, moving the position h is equivalent to moving the boundary of the small hole with a uniform speed (the shape of the hole remains constant). However, the formulas for the derivatives of $\mathscr{J}_\varepsilon$ have been derived easily and the numerical application is also less involved than for the usual speed method setting, where the boundary of the hole needs to be parameterized.

6.1 The Discrete Algorithm

The usual Fletcher-Reeves algorithm is as follows. At the step $k+1$ the point h_k becomes

$$h_{k+1} = h_k + t_k d_k, \tag{82}$$

where d_k is the direction of descent given by

$$d_k = -\nabla_h \mathscr{J}(h_k) + \frac{\nabla_h \mathscr{J}(h_k)^T \nabla_h \mathscr{J}(h_k)}{\nabla_h \mathscr{J}(h_{k-1})^T \nabla_h \mathscr{J}(h_{k-1})} d_{k-1}, \tag{83}$$

and the time step t_k is given by a line-search procedure.

Algorithm: In the subsequent algorithm subscript l, for all quantities refers to the discrete counterpart of the respective continuous variable, while the superscript k refers to the k-th iteration of the algorithm. The discretization is based on finite differences. We assume that the discretized domain is given by a uniform grid with mesh size l. We denote the grid points by $\mathbf{x}_i, i = 1,..,N$. For the discretization of the

Laplace operator we use the standard five points stencil. The grid functions $v_l^0, G_l, \ldots$ are defined on the grid points.

Step 1 Set $k = 0$. Compute $v_l^{0,k}$ from (4)-(5) and the topological derivative $\mathcal{T}_l^k(h)$ at every $h \in \Omega_1$. Deduce a starting point h_0 by taking $h_0 = \mathrm{argmin}_{h\in\Omega_1} \mathcal{T}_l^k(h)$. Set a tolerance γ.
Step 2 Compute $G_l^k, \mathcal{G}_l^k$ from the discrete relaxed system corresponding to (8)-(9) and (11)-(12) and deduce $\lambda_l^k(h)$ and $\mathbf{v}_l^k$. Evaluate the cost functional $J_{\varepsilon,l}^k(h^k)$.
Step 3 If the direction d^k verifies $\|d^k\| < \gamma l(1+d^0)$ and $\|h^k - h^{k-1}\| < \gamma l$ then stop; otherwise continue with step 4.
Step 4 Update h^k. Put $k := k+1$. Go to step 2.

7 Numerical Example

Several examples are presented here, with different source terms f. The domain Ω is taken as the square $\Omega = [0,1]\times[0,1]$. For each example, the number of iterations are given, the initial value for h (given by the topological derivative) and the final value of h are compared to the real position of the hole. We also give a plot for the grid 512×512 of the convergence of the functional $\mathcal{J}_\varepsilon(h)$, of the measured data z and of the reconstructed solution $\mathbf{v}_{h_f}$.

The observation z is artificial, which means that we know *ad hoc* the location of the hole h^* but we start the procedure from another value of h. In order to compute this observation z precisely, we use finite differences with a Shortley-Weller approximation [5] to discretize the Laplacian on the boundary of Ω_ε. The position h^* denotes the real center of the hole. The set ω is chosen as a ball of radius 1. The notation h_0 and h_f denote the initial and final value of h, respectively. In every example we take $\Omega_1 = B(h^*, 0.4)$, but h^* takes different values. In what follows, (x_1,x_2) denotes the Cartesian coordinates in $\mathbb{R}^2$.
Example 1. In the first example, the data is $f \equiv 70$ and $h^* = (0.5,0.5)$. Due to the very simple and symmetric situation, the topological derivative is enough to find the position of the hole and no further iterations are needed after the initialization. Results are presented in Table 1.

Table 1 Example 1

l	iterations	h_0	h_f	h^*
$1/128$	1	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
$1/256$	1	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
$1/512$	1	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)

Example 2. In the second example, the data is $f(x_1,x_2) = 100x_1^2x_2 + 10$ and $h^* = (0.4,0.4)$. One can see in this example that the topological derivative (which

gives h_0) was far from the optimal solution while our algorithm converges to the true solution as $l \to 0$. The corresponding figure is Fig. 2. Results are presented in Table 2.

Table 2 Example 2

l	iterations	h_0	h_f	h^*
$1/128$	11	(0.6953,0.6640)	(0.3750,0.3570)	(0.4,0.4)
$1/256$	12	(0.6953,0.6679)	(0.3913,0.3828)	(0.4,0.4)
$1/512$	14	(0.6933,0.6718)	(0.3965,0.3959)	(0.4,0.4)

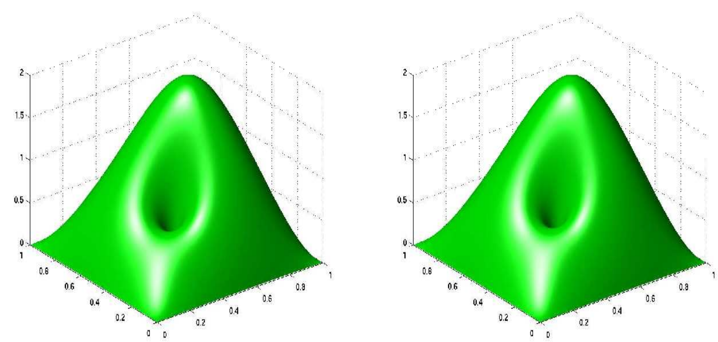

Fig. 2 Solution $\mathbf{v}_{h_f}$ (left) and data z (right)

Example 3. In the third example, the data is $f(x_1,x_2) = 100\sin(4\pi x_1)\sin(4\pi x_2) + 10$ and $h^* = (0.6, 0.6)$. The topological derivative gives again an initialization far from the optimal solution while our algorithm converges to the true solution as $l \to 0$. The corresponding figure is Fig. 3. Results are presented in Table 2.

Table 3 Example 3

l	iterations	h_0	h_f	h^*
$1/128$	29	(0.3593, 0.3593)	(0.6405, 0.6405)	(0.6,0.6)
$1/256$	23	(0.3593 , 0.3593)	(0.6214, 0.6214)	(0.6,0.6)
$1/512$	21	(0.3613,0.3613)	(0.6055,0.6055)	(0.6,0.6)

Example 4. In the fourth example, the data is

$$f(x_1,x_2) = -(60\cos(4\pi r) + 30)\mathbb{1}_{\{r<0.4\ \&\ r>0.1\}}$$

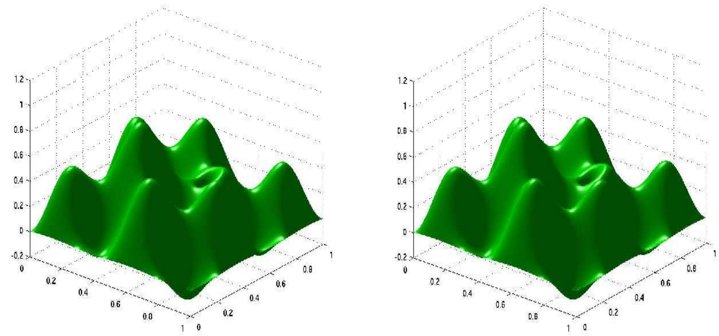

Fig. 3 Solution $\mathbf{v}_{h_f}$ (left) and data z (right)

with $r = |(x_1, x_2) - h^*|$ and $h^* = (0.6, 0.6)$. The corresponding figure is Fig. 4. Results are presented in table 4. Similar conclusions as for the previous examples are drawn.

Table 4 Example 4

l	iterations	h_0	h_f	h^*
$1/128$	63	(0.4296,0.4296)	(0.6477,0.6477)	(0.6,0.6)
$1/256$	53	(0.4296,0.4257)	(0.6195 ,0.6202)	(0.6,0.6)
$1/512$	44	(0.4277,0.4277)	(0.6042,0.6042)	(0.6,0.6)

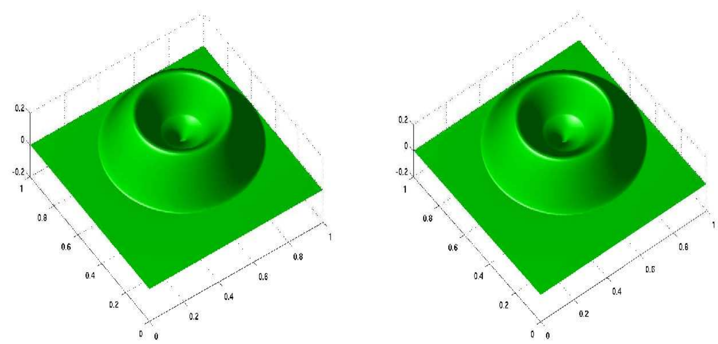

Fig. 4 Solution $\mathbf{v}_{h_f}$ (left) and data z (right)

8 Conclusion

In the numerical examples above, the experimental data z is given on approximately 50% of the domain. Except for the simple first example, the topological derivative is not able to find the correct position of the hole, while our algorithm does.

Self-adjoint extensions of elliptic operators are not restricted to Dirichlet conditions, and can be used with other boundary conditions, including those of Neumann type. Further research will be focused on applying this modeling to evolution boundary problems and spectral problems.

In this paper, we restricted ourselves to the case of a small ball for the numerical applications, but the general case of any shape can be derived easily. The case of several holes is also interesting, and requires additional work for the theoretical background. Indeed, in the expansion of the solution, the holes interact with each other and the expansion is more involved. Such an analysis can be found, e.g. in [14].

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