Free Material Optimization for Plates and Shells

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Abstract In this article, we present the Free Material Optimization (FMO) problem for plates and shells based on Naghdi's shell model. In FMO – a branch of structural optimization – we search for the ultimately best material properties in a given design domain loaded by a set of given forces. The optimization variable is the full material tensor at each point of the design domain. We give a basic formulation of the problem and prove existence of an optimal solution. Lagrange duality theory allows to identify the basic problem as the dual of an infinite-dimensional convex nonlinear semidefinite program. After discretization by the finite element method the latter problem can be solved using a nonlinear SDP code. The article is concluded by a few numerical studies.

1 Introduction

Structural optimization deals with the problem of finding the stiffest structure subjected to a set of given loads and boundary conditions, when only a limited amount of material resources is available. Nowadays, this approach plays an important role in the construction of light-weight structures like airplanes and cars. Large parts of these objects as, for instance, the fuselage, consist of thin-walled structures like shells and plates. This is the reason why structural optimization of shells has received a lot of attention in the design optimization community over the last couple of years. For example, shape optimization techniques have been used to vary the geometry and boundary of a shell with the goal to stiffen the structure [6]. Various approaches try to identify the optimal topology of a shell in the sense of 0–1–designs.

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For an overview in the case of plates see [4]. On spherical shells it is possible to calculate the topological derivative and to exploit this information with the purpose of finding the optimal position of holes [17]. Only recently, free sizing optimization taking strength and stability constraints into account has been used to improve the design of shell structures [7].

Another important class of shell design problems is based on material optimization. Here the design variables reflect not only the distribution of material in the design domain, but also the local properties of the material. The methods used in the area of material optimization differ in the choice of the admissible set of materials. In [18] a pseudo density of the material is varied using a SIMP-approach. Rather than cutting the solution space down to 0–1–designs the author proposes to realize the optimal solution using foams that can be produced in manifold densities. In aerospace industry the use of composite materials is very common. In [23] the authors suggest to design composite shells by optimization of the material selection and fibre angles in a laminated shell structure. It is even possible to consider fully anisotropic elasticity tensors as admissible set for the design optimization as shown for Reissner-Mindlin plates in [2]. Finally, there are approaches taking advantage of adaptive methods – either by changing the parametrization of the design space during optimization or by adapting the model via switching between shape and material optimization; see [20].

In this article, we focus on Free Material Optimization, originally introduced for the optimal design of solid bodies by [3]. The design variable used in Free Material Optimization is the full material tensor at each point of the design domain. Therefore it yields not only the optimal material distribution, but also the material properties at each point. Various solution techniques for this problem have been proposed; see, for example, [25]. Due to the high freedom in the design space the resulting material/structure is typically hard to manufacture. Nevertheless it gives valuable information about the optimal material density, symmetry and principal directions, which can be exploited to realize approximations of the optimal design. One possible realization by tapelayering is described in [13]. In the recent years, the formulation of the Free Material Optimization problem has been extended to cover multiple load cases [1], stability control by consideration of global buckling [15] and stress constraints [16]. In this article we propose a formulation of Free Material Optimization based on the linear elastic shell model of Naghdi [21] suited for thin-walled structures like airplanes, cars and pipes.

2 Naghdi's Shell Model

We start with a mathematical description of Naghdi's shell model using the standard notation e.g. described in [21, 8, 9]. The geometry of a Naghdi shell is described by the midsurface ω – an open bounded two-dimensional set in Euclidean space, which can be parametrized by a sufficiently smooth function $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $\Phi \in W^{2,\infty}(\omega)$. This is in contrast to other popular shell models as for example the Kirchhoff-Love model, where one starts from a three-dimensional solid material and makes approximations accounting to the thinness of the shell.

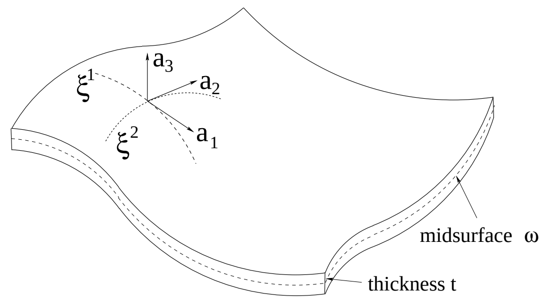

Fig. 1 Curvilinear coordinates on the midsurface

Hence it is advantageous to use curvilinear coordinates denoted by ξ^i (in accordance with common notation in shell theory Latin indices run over 1, 2 and 3, while Greek indices run only over 1 and 2). The covariant basis vectors are then defined by

$$a_\alpha = \frac{\partial \Phi}{\partial \xi^\alpha}, \qquad a_3 = \frac{a_1 \times a_2}{\|a_1 \times a_2\|}. \tag{1}$$

Moreover, the surface covariant derivative of a vector field v is given by

$$v_{\alpha|\mu} = v_{\alpha,\mu} - \Gamma^\lambda_{\alpha\mu} v_\lambda, \tag{2}$$

where $v_{\alpha,\mu}$ is the partial derivative of v_α with respect to ξ^μ and $\Gamma^\lambda_{\alpha\mu}$ is the Christoffel symbol of the midsurface

$$\Gamma^\lambda_{\alpha\mu} = a_{\alpha,\mu} \cdot a^\lambda. \tag{3}$$

Furthermore the fundamental forms of the midsurface are defined by

- first fundamental form: $a_{\alpha\beta} = a_\alpha \cdot a_\beta$,
- second fundamental form: $b_{\alpha\beta} = -a_{3,\beta} \cdot a_\alpha$,
- third fundamental form: $c_{\alpha\beta} = b^\lambda_\alpha b_{\lambda\beta}$.

It turns out that the midsurface alone contains not enough information to describe bending and shear effects. A remedy is provided by the theory of Cosserat continua: at each point $x \in \omega$ a director vector d is attached to the shell, adding the lacking degrees of freedom [10, 22]. These director vectors can be interpreted as material lines along the thickness of the shell. The deformation of the loaded shell can be described by a translation of all points of the midsurface $u \in \left[H^1(\omega)\right]^3$ and a rotation of the associated director vectors stemming from the group $SO(2)$. As the rotation of an infinitely-thin straight material line is uniquely defined by a rotation vector normal to that line we introduce $\theta \in \left[H^1(\omega)\right]^2$ to represent the rotation by $\theta_\lambda a^\lambda$ [8]. A component on a_3 is not required due to the fact that rotations of the director vectors around their own axis are neglected. Thus we obtain the following displacement formula:

$$U(\xi^1,\xi^2,\xi^3) = u(\xi^1,\xi^2) + \xi^3 \theta_\lambda(\xi^1,\xi^2) a^\lambda(\xi^1,\xi^2). \tag{4}$$

In the remainder of this article, we consider a shell with a Lipschitz boundary $\partial\omega$. The shell is clamped at parts of the boundary. To this end we partition $\partial\omega$ into two sets $\partial\omega_0$ and $\partial\omega_1$ which are open in $\partial\omega$, $\partial\omega = \overline{\partial\omega}_0 \cup \overline{\partial\omega}_1$ and $\partial\omega_0 \cap \partial\omega_1 = \emptyset$. Then Dirichlet boundary conditions are applied on $\partial\omega_0$ and the shell is subjected to forces and moments on $\partial\omega_1$. Using this, we define the set of admissible displacements to be

$$\mathscr{U} := \{ (u,\theta) \in [H^1(\omega)]^5 \big| u = 0 \text{ and } \theta = 0 \text{ on } \partial\omega_0 \}. \tag{5}$$

As a consequence we obtain $\left[H_0^1(\omega)\right]^5 \subset \mathscr{U} \subset \left[H^1(\omega)\right]^5$. It is now possible to deduce formulas for membrane strains $\gamma_{\alpha\beta}$, bending strains $\chi_{\alpha\beta}$ and shear strains ζ_α, respectively:

$$\begin{aligned}
\gamma_{\alpha\beta}(u) &= \frac{1}{2}\left(u_{\alpha|\beta} + u_{\beta|\alpha}\right) - b_{\alpha\beta} u_3, \\
\chi_{\alpha\beta}(u,\theta) &= \frac{1}{2}\left(\theta_{\alpha|\beta} + \theta_{\beta|\alpha} - b^\lambda_\beta u_{\lambda|\alpha} - b^\lambda_\alpha u_{\lambda|\beta}\right) + c_{\alpha\beta} u_3, \\
\zeta_\alpha(u,\theta) &= \frac{1}{2}\left(\theta_\alpha + u_{3,\alpha} + b^\lambda_\alpha u_\lambda\right).
\end{aligned} \tag{6}$$

The assumption of linear elasticity in Naghdi's shell model leads to the following Hooke's law:

$$\begin{aligned}
N^{\lambda\mu} &= t\, C^{\lambda\mu\alpha\beta} \gamma_{\alpha\beta}, \\
M^{\lambda\mu} &= \frac{t^3}{12} C^{\lambda\mu\alpha\beta} \chi_{\alpha\beta}, \\
m^\lambda &= t\, D^{\lambda\alpha} \zeta_\alpha.
\end{aligned} \tag{7}$$

Here $C^{\lambda\mu\alpha\beta}$ and $D^{\lambda\alpha}$ are the elasticity tensors of the shell. $C^{\lambda\mu\alpha\beta}$ is a fourth-order tensor with the following symmetries:

$$C^{\lambda\mu\alpha\beta} = C^{\mu\lambda\alpha\beta}, \quad C^{\lambda\mu\alpha\beta} = C^{\lambda\mu\beta\alpha}, \quad C^{\lambda\mu\alpha\beta} = C^{\alpha\beta\lambda\mu}. \tag{8}$$

$D^{\lambda\alpha}$ is a symmetric second order tensor satisfying $D^{\lambda\alpha} = D^{\alpha\lambda}$. Moreover, the symmetric second order tensors $N^{\lambda\mu}$ and $M^{\lambda\mu}$ are called force resultant and moment resultant, respectively, and m^λ is the transverse shear force resultant. Finally t is the thickness of the shell. In the following we assume the thickness of the shell to be constant. Note however that the main results presented in this article remain valid for a thickness profile $t = t(x)$, which remains unchanged during optimization. The symmetry of the tensors allows us to rewrite Hooke's law using the following vectors and matrices:

$$\gamma = \begin{pmatrix} \gamma_{11} \\ \gamma_{22} \\ \sqrt{2}\gamma_{12} \end{pmatrix}, \qquad \chi = \begin{pmatrix} \chi_{11} \\ \chi_{22} \\ \sqrt{2}\chi_{12} \end{pmatrix}, \qquad \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \tag{9}$$

$$N=\begin{pmatrix} N_{11} \\ N_{22} \\ \sqrt{2}N_{12} \end{pmatrix}, \qquad M=\begin{pmatrix} M_{11} \\ M_{22} \\ \sqrt{2}M_{12} \end{pmatrix}, \qquad m=\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \tag{10}$$

$$C=\begin{pmatrix} C_{1111} & C_{1122} & \sqrt{2}C_{1112} \\ C_{1122} & C_{2222} & \sqrt{2}C_{2212} \\ \sqrt{2}C_{1112} & \sqrt{2}C_{2212} & 2C_{1212} \end{pmatrix}, \qquad D=\begin{pmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{pmatrix}. \tag{11}$$

Then Hooke's law takes the form

$$\begin{aligned} N(x) &= t\,C(x)\gamma(u(x)), \\ M(x) &= \frac{t^3}{12}C(x)\chi(u(x),\theta(x)), \\ m(x) &= t\,D(x)\zeta(u(x),\theta(x)) \end{aligned} \tag{12}$$

and the potential energy $\Pi(u,\theta)$ of the Naghdi shell can be written as

$$\begin{aligned} \Pi(u,\theta) = &\frac{1}{2}\int_\omega \left(t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta\right) dS \\ &- \int_\omega t f^\top u\, dS - \int_{\partial\omega_1} \left(g_u^\top u + g_\theta^\top \theta\right) dl, \end{aligned} \tag{13}$$

where $f \in \left[L^2(\omega)\right]^3$ is a given force resultant density and $g_u \in \left[L^2(\partial\omega_1)\right]^3$ and $g_\theta \in \left[L^2(\omega)\right]^2$ are given traction and moment resultant densities, respectively. The shell is in equilibrium for any $(u,\theta) \in \mathscr{U}$ that minimizes the potential energy

$$\min_{(u,\theta)\in\mathscr{U}} \Pi(u,\theta). \tag{14}$$

It is also possible to treat plates in this context. Assuming a planar midsurface allows to deduce the Reissner-Mindlin plate model from Naghdi's shell model. A planar midsurface has no curvature and thus a constant normal vector a_3. This results in vanishing second and third fundamental forms of the midsurface ω:

$$b_{\alpha\beta} = 0, \quad c_{\alpha\beta} = 0. \tag{15}$$

In this case the formulas for the strains boil down to:

$$\begin{aligned} \gamma_{\alpha\beta}(u_1,u_2) &= \frac{1}{2}\left(u_{\alpha|\beta} + u_{\beta|\alpha}\right), \\ \chi_{\alpha\beta}(\theta) &= \frac{1}{2}\left(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}\right), \\ \zeta_\alpha(u_3,\theta) &= \frac{1}{2}\left(\theta_\alpha + u_{3,\alpha}\right). \end{aligned} \tag{16}$$

The equilibrium state of the plate is again found by minimizing the potential energy

$$\min_{(u,\theta)\in\mathscr{U}} \Pi(u,\theta) = \frac{1}{2}\int_\omega \Big(t\gamma^\top(u_1,u_2)C\gamma(u_1,u_2) + \frac{t^3}{12}\chi^\top(\theta)C\chi(\theta) \\ + t\zeta^\top(u_3,\theta)D\zeta(u_3,\theta)\Big)\, dS - \int_\omega t f^\top u\, dS - \int_{\partial\omega_1} \Big(g_u^\top u + g_\theta^\top \theta\Big)\, dl. \tag{17}$$

When solving the elasticity problem for a plate this can be separated into the membrane problem

$$\min_{(u_1,u_2)\in\mathscr{U}} \frac{1}{2}\int_\omega t\gamma^\top(u_1,u_2)C\gamma(u_1,u_2)\, dS - \int_\omega t\,(f_1u_1 + f_2u_2)\, dS \\ - \int_{\partial\omega_1} (g_{u1}u_1 + g_{u2}u_2)\, dl \tag{18}$$

and the so-called Reissner-Mindlin problem

$$\min_{(u_3,\theta)\in\mathscr{U}} \frac{1}{2}\int_\omega \left(\frac{t^3}{12}\chi^\top(\theta)C\chi(\theta) + t\zeta^\top(u_3,\theta)D\zeta(u_3,\theta)\right) dS \\ - \int_\omega t f_3 u_3\, dS - \int_{\partial\omega_1} \Big(g_{u3}u_3 + g_\theta^\top\theta\Big)\, dl. \tag{19}$$

3 The Single Load Problem

Up to now we have merely described the physical behavior of the shell. However our overall goal is to find the stiffest structure which is subjected to a given set of loads f, g_u and g_θ. A measure on how much a structure will deform under these loads is given by the compliance

$$\begin{aligned} \mathrm{comp}(C,D) &= -\min_{(u,\theta)\in\mathscr{U}} 2\,\Pi_{C,D}(u,\theta) = \max_{(u,\theta)\in\mathscr{U}} -2\,\Pi_{C,D}(u,\theta) \\ &= \max_{(u,\theta)\in\mathscr{U}} -\int_\omega \left(t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta\right) dS \\ &\quad + 2\int_\omega t f^\top u\, dS + 2\int_{\partial\omega_1} \Big(g_u^\top u + g_\theta^\top\theta\Big)\, dl. \end{aligned} \tag{20}$$

Apparently the compliance is given by twice the negative potential energy in equilibrium. In order to find the stiffest structure possible we now minimize the compliance with respect to the design variables. As we intend to work with Free Material Optimization these variables are the full elasticity tensors C and D. We want to allow for holes and material-no-material situations in the optimal structures, therefore we choose $C \in [L^\infty(\omega)]^{3\times 3}$ and $D \in [L^\infty(\omega)]^{2\times 2}$. As pointed out in Section 2 the matrices have to be symmetric, furthermore they also have to be positive semidefinite as they describe a physical material:

$$C = C^\top \succeq 0, \quad D = D^\top \succeq 0. \tag{21}$$

As a measure for the amount of material used at a certain point $x \in \omega$ we simply use the summed traces of the matrices $t\left(\mathrm{tr}(C) + \frac{1}{2}\mathrm{tr}(D)\right)$. The factor $\frac{1}{2}$ is necessary to be able to compare the results with the three–dimensional solid case. As we want to limit the material resources, we add the volume constraint

$$\int_\omega t\left(\mathrm{tr}(C) + \tfrac{1}{2}\mathrm{tr}(D)\right)\, dS \le V. \tag{22}$$

Finally we add box constraints to avoid arbitrarily high material concentrations at single points:

$$0 \le \rho^- \le t\left(\mathrm{tr}(C) + \tfrac{1}{2}\mathrm{tr}(D)\right) \le \rho^+. \tag{23}$$

Summarizing (21), (22) and (23) we obtain the set of admissible elasticity tensors

$$\mathscr{C} := \left\{ (C,D) \in [L^\infty(\omega)]^{3\times 3} \times [L^\infty(\omega)]^{2\times 2} \;\middle|\; \begin{array}{l} C = C^\top \succeq 0 \\ D = D^\top \succeq 0 \\ \int_\omega t(\mathrm{tr}(C(x)) + \frac{1}{2}\mathrm{tr}(D(x)))\, dS \le V \\ 0 \le \rho^- \le t(\mathrm{tr}(C) + \frac{1}{2}\mathrm{tr}(D)) \le \rho^+ \end{array} \right\} \tag{24}$$

For simplicity of notation we will assume $\rho^- = 0$. But note that all statements presented in this paper are also true for positive ρ^-. We finally are able to formulate the single load problem for shells, in which we seek the design variables C and D which yield the minimal compliance:

$$\begin{aligned} \min_{(C,D)\in\mathscr{C}} \max_{(u,\theta)\in\mathscr{U}} & -\frac{1}{2}\int_\omega \left(t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta \right) dS \\ & + \int_\omega t f^\top u\, dS + \int_{\partial\omega_1} \left(g_u^\top u + g_\theta^\top \theta \right) dl. \end{aligned} \tag{25}$$

Introducing the function

$$\begin{aligned} J((C,D),(u,\theta)) := & -\frac{1}{2}\int_\omega \left(t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta \right) dS \\ & + \int_\omega t f^\top u\, dS + \int_{\partial\omega_1} \left(g_u^\top u + g_\theta^\top \theta \right) dl \end{aligned} \tag{26}$$

we rewrite the latter optimization problem as

$$\min_{(C,D)\in\mathscr{C}} \max_{(u,\theta)\in\mathscr{U}} J((C,D),(u,\theta)). \tag{27}$$

In the case of plates we start from the equilibrium problem (17). The uncoupling into the membrane and the Reissner–Mindlin problem is not possible anymore when working with Free Material Optimization, as the material tensor C is one of the optimization variables connecting the membrane and bending terms. Thus the single load problem for Reissner–Mindlin plates takes the form

$$\min_{(C,D)\in\mathscr{C}} \max_{(u,\theta)\in\mathscr{U}} J((C,D),(u,\theta)) := -\frac{1}{2}\int_\omega \left(t\gamma^\top C\gamma + \frac{t^3}{12}\chi^\top C\chi + t\zeta^\top D\zeta \right) dS + \int_\omega t f^\top u\, dS + \int_{\partial\omega_1} \left(g_u^\top u + g_\theta^\top \theta \right) dl. \tag{28}$$

This problem has already been formulated by Bendsøe and Díaz, who propose a solution via analytic derivation of the optimal material properties [2].

We now want to show existence of optimal solutions for problem (27). It can be easily seen that an optimal solution of the single load problem for shells is a saddle-point of the functional $J((C,D),(u,\theta))$. Thus existence of an optimal point follows from a Minimax-Theorem.

Theorem 1. *Problem (27) has an optimal solution* $((C^*,D^*),(u^*,\theta^*)) \in \mathscr{C}\times\mathscr{U}$.

The proof uses the modified Minimax-Theorem presented in [19] that allows for $\mathscr{C}$ to be subset of the dual of a non–reflexive Banach–space – in this case $L^1(\omega)$. The required ellipticity of Naghdi's shell model has been shown in [11, 5]. The complete proof can be found in [12].

4 The Primal Problem

In [24] it has been shown that the Free Material Optimization problem for solid material can be transformed into a linear quadratically constrained optimization problem using duality theory. During this section we show that a similar technique can be applied on the Free Material Optimization problem for shells resulting in a convex nonlinear semidefinite program instead of the saddle-point problem given in (27).

Theorem 2. *Problem (27) is equivalent to the Lagrange dual problem of*

$$\begin{aligned} \max_{\substack{(u,\theta)\in\mathscr{U}\\ \alpha\in\mathbb{R}_0^+\\ \beta_{u,l}\in L^1(\omega)\\ \beta_{u,l}\geq 0}} \quad & \int_\omega t f^\top u\, dS + \int_{\partial\omega_1} (g_u^\top u + g_\theta^\top\theta) dl - \alpha V - \rho^+ \int_\omega \beta_u dS \\ \text{subject to} \quad & \frac{t}{2}\gamma(u)\gamma(u)^\top + \frac{t^3}{24}\chi(u,\theta)\chi(u,\theta)^\top - t(\alpha+\beta_u-\beta_l)E_3 \preceq 0 \\ & \frac{t}{2}\zeta(u,\theta)\zeta(u,\theta)^\top - \frac{t}{2}(\alpha+\beta_u-\beta_l)E_2 \preceq 0 \end{aligned} \tag{29}$$

where E_n *denotes the unit matrix in* $\mathbb{R}^n$.

In order to prove this theorem we construct the Lagrangian to problem (29). It can then be shown that this problem is equivalent to the original problem (27). The proof follows the ideas presented in [24, Theorem 3.3.4] and is given in detail in [12].

Problem (29) is a convex nonlinear semidefinite program (SDP). Compared to the original problem formulation (27) problem (29) has several advantages. The matrices C and D are hidden in the problem as Lagrange multipliers. This significantly reduces the number of variables in the discrete problem (compare Section 5). Furthermore problem (29) is convex. As t is the thickness of the shell, it is strictly positive and the matrix constraints of (29) can be simplified to

$$\begin{aligned} \gamma(u)\gamma(u)^\top + \frac{t^2}{12}\chi(u,\theta)\chi(u,\theta)^\top - 2(\alpha+\beta_u-\beta_l)E_3 \preceq 0\,, \\ \zeta(u,\theta)\zeta(u,\theta)^\top - (\alpha+\beta_u-\beta_l)E_2 \preceq 0\,. \end{aligned} \tag{30}$$

5 Numerical Treatment

5.1 Discretization

We now intend to solve the infinite-dimensional SDP (29) numerically. For this purpose we discretize the problem by the finite element method [8]. The midsurface ω is partitioned into M elements ω_m. The number of corresponding element nodes is denoted by n. The elasticity matrices $C(x)$ and $D(x)$ are approximated by elementwise constant matrices $(C_1,\ldots,C_M)$ and $(D_1,\ldots,D_M)$ where $C_m \in \mathbb{R}^{3\times3}$ and $D_m \in \mathbb{R}^{2\times2}$ for all $m = 1,\ldots,M$. The displacements take the following form

$$U^{3D} = \sum_{i=1}^{n} \lambda_i(r,s)\left(u^{(i)} + z\frac{t}{2}\,\theta^{(i)}\right)\,, \tag{31}$$

where the $\lambda_i(r,s)$ are bilinear 2D Lagrange shape functions. This assures that the Reissner-Mindlin assumption – material lines remain straight and unstretched during deformation – is fulfilled at all nodes of the mesh. Using (31) the discretized membrane strain matrix B_i^γ becomes

$$B_i^\gamma = \begin{pmatrix} \lambda_{i|1} & 0 & -b_{11}\lambda_i & 0 & 0 \\ 0 & \lambda_{i|2} & -b_{22}\lambda_i & 0 & 0 \\ \frac{1}{\sqrt{2}}\lambda_{i|2} & \frac{1}{\sqrt{2}}\lambda_{i|1} & -\sqrt{2}\,b_{12}\lambda_i & 0 & 0 \end{pmatrix} \tag{32}$$

The factor $\sqrt{2}$ stems from the vector-matrix-notation introduced in (9). Using this, the discrete counterpart of the dyadic product $\gamma\gamma^\top$ reads

$$A_m^\gamma(u) = \sum_{i,j\in K} \int_{\omega_m} B_j^\gamma U U^\top (B_i^\gamma)^\top dx\,, \tag{33}$$

where K is the index set of nodes associated with the element m. Analogously we derive

$$A_m^{\chi}(u) = \sum_{i,j\in K} \int_{\omega_m} B_j^{\chi} U U^\top (B_i^{\chi})^\top dx\,, \tag{34}$$

$$A_m^{\zeta}(u) = \sum_{i,j\in K} \int_{\omega_m} B_j^{\zeta} U U^\top (B_i^{\zeta})^\top dx\,. \tag{35}$$

Replacing the forces and moments in problem (29) by their discrete counterparts we get the following discrete single load FMO problem for shells:

$$\begin{aligned}
\max_{\substack{(u,\theta)\in\mathscr{U}\\ \alpha\in\mathbb{R}_0^+\\ \beta_u,\beta_l\in\mathbb{R}_0^{+M}}} \quad & \sum_{i=1}^{n} \left(t f_i u_i - \rho^+ \beta_{u_i}\right) + \sum_{i\in\partial\omega_1} (g_{u_i} u_i + g_{\theta_i}\theta_i) dl - V\alpha \\
\text{subject to} \quad & \frac{t}{2} A_m^{\gamma}(u) + \frac{t^3}{24} A_m^{\chi}(u,\theta) - t(\alpha + \beta_u - \beta_l) E_3 \preceq 0 \\
& \frac{t}{2} A_m^{\zeta}(u,\theta) - \frac{t}{2}(\alpha + \beta_u - \beta_l) E_2 \preceq 0\,.
\end{aligned} \tag{36}$$

Obviously (36) is a finite-dimensional convex nonlinear semidefinite program.

5.2 Examples

Two numerical examples are presented in this section. In order to solve problem (36) we have used the nonlinear SDP code PENNON [14]. Although only the resulting "density" function $t\,\mathrm{tr}(C) + \frac{1}{2}t\,\mathrm{tr}(D)$ is depicted, we want to emphasize that the code provides the optimal elasticity matrices C_m and D_m for each element ω_m, $m = 1,\ldots,M$. Thus we gather information about the optimal material symmetry and material directions usable in the manufacturing process.

Example 1. The first example (Fig. 2) serves as a test for the consistency with the two-dimensional solid case. We consider a rectangular plate with in-plane forces. The plate is clamped on one side while forces are applied in the center of the opposite edge and directed in parallel to the boundary. This example known as Michell truss is widely used in topology optimization literature.

The typical material distribution of a Michell truss can be easily recognized in the displayed "density" distribution (Fig. 3). It is also notable that only membrane strains appear as there is no deformation outside the midsurface. Thus there is no material used for the matrix D which accounts to shear effects.

Example 2. The second example employs all degrees of freedom of the shell. We start with a saddle-shaped midsurface, that is clamped on one side (Fig. 4). A vertical force acts in the center of the opposite edge. This example can be interpreted as optimization of a coat hook fixed to the wall.

The resulting "density distribution" (Fig. 5) shows a firm tip at the location of the load. The shell tries to avoid vertical bending and distributes material over the

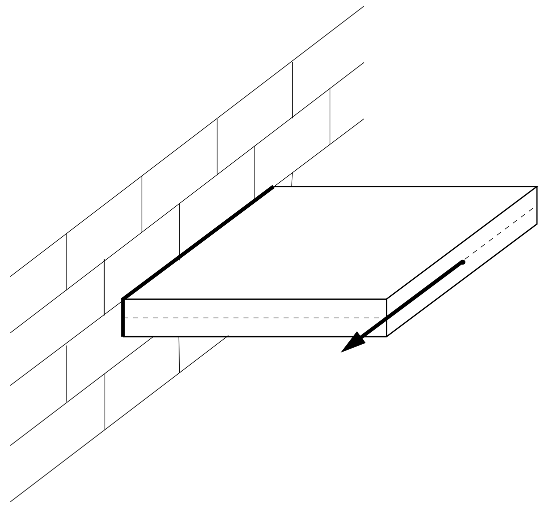

Fig. 2 Michell truss load case

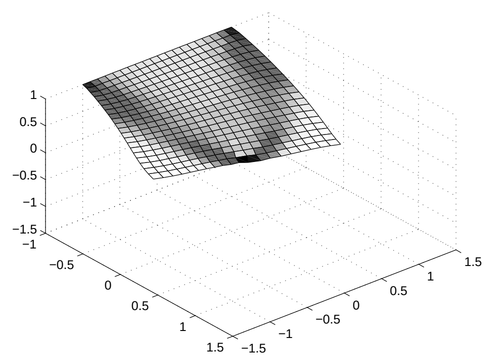

Fig. 3 Michell truss "density distribution"

complete design domain (apart from the corners in the front which are not suited to stiffen this particular structure). No holes can thus be found in contrast to the previous example. This result is not unexpected as stiff triangle-shaped structures appearing in the plane of loading are well known in topology optimization.

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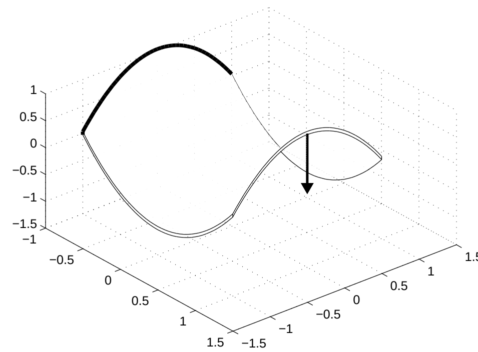

Fig. 4 Saddle-shaped hook load case

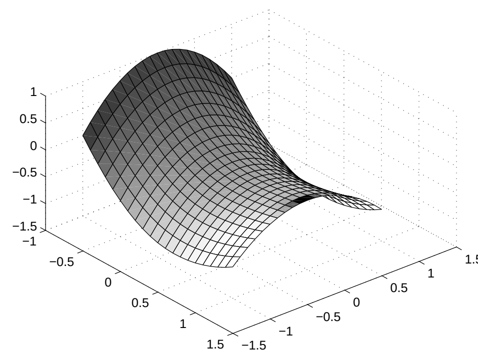

Fig. 5 Saddle-shaped hook "density distribution"

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