

Galerkin Strategy for Level Set Shape Analysis: Application to Geodesic Tube

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Abstract In this paper, we consider the geodesic tube characterization using a Galerkin-Level Set strategy. The first section is devoted to the analysis of a geodesic tube construction between two sets through the definition of the shape metric. In the second section, we define the Galerkin-Level Set strategy in shape analysis. This new variational formulation associated to a Hilbert space metric for shape identification problem consists in parameterizing the level set function in a finite dimensional subspace spanned by linear independent functions. Consequently, this method is more focused on topological changes than on high accuracy for the boundary evaluation as in a traditional level set formulation. In the third section, we use the Galerkin-Level Set formulation applied to a geodesic tube construction between two sets, through the calculus of the shape derivative of the normal speed. Finally, this geodesic tube construction is validated by a numerical experiment.

1 Tube Formulation using Moving Domain

In this section, we briefly recall the concept of connecting tube, introduced in [6]. Let us consider \mathbb{D} as a bounded universe in \mathbb{R}^n and two open sets domains $\Omega_0, \Omega_1 \subset \mathbb{D}$. We denote the initial domain by Ω_0 and the final domain by Ω_1 , and consider the tube connecting Ω_0 with Ω_1 defined by the $n + 1$ dimensional graph of an n -dimensional moving domain: see Fig. 1. Consequently, considering the time interval $I = [0, 1]$, we define the tube evolution Q by product space, using the cylinder $I \times \Omega$ as follows:

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$$Q = \bigcup_{0 \leq t \leq 1} \{t\} \times \Omega_t \quad (1)$$

Moreover, we denote by Σ the lateral boundary of the tube, defined by the following expression: $\Sigma = \bigcup_{0 \leq t \leq 1} \{t\} \times \Gamma_t$, where Γ_t denotes the boundary of Ω_t . The characteristic function of the tube is defined by $\zeta(t, x) \stackrel{\text{def}}{=} \chi_{\Omega_t}(x)$ and verifies $\zeta^2 = \zeta$. Following [4, 5], the set of connecting tubes between Ω_0 and Ω_1 is defined by:

$$\mathcal{T}(\Omega_0, \Omega_1) = \left\{ \zeta \in L^\infty(I \times \mathbb{D}) \text{ and piecewise } C^1, \begin{cases} \zeta(0) = \chi_{\Omega_0} \\ \zeta(1) = \chi_{\Omega_1} \end{cases} \right\} \quad (2)$$

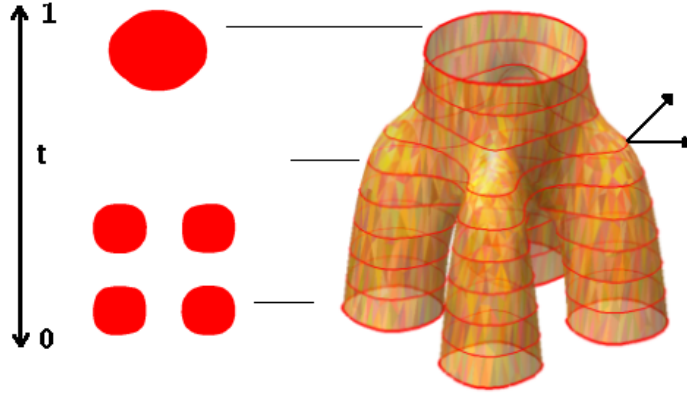


Fig. 1 Continuous tube between Ω_0 and Ω_1 .

The outgoing unitary normal vector field on the lateral boundary of the tube Σ is defined by

$$v(t, x) = \frac{1}{\sqrt{1 + v(t, x)^2}} \begin{pmatrix} -v(t, x) \\ \mathbf{n}(t, x) \end{pmatrix} \quad (3)$$

where $\mathbf{n}(t, x)$ is the normal field to Γ_t and $v(t, x)$ is an intrinsic geometric entity called the normal speed of the boundary Γ_t .

Definition 1. In order to characterize the minimal tube path between Ω_0 and Ω_1 , we introduce the function:

$$d(\Omega_0, \Omega_1) = \inf_{\zeta \in \mathcal{T}(\Omega_0, \Omega_1)} \int_0^1 \int_{\Gamma_t} |v(t, x)| d\Gamma(x) dt \quad (4)$$

Lemma 1. The function $d(\Omega_0, \Omega_1)$ is a metric.

Proof. We have to prove that the function $d(\Omega_0, \Omega_1)$ satisfies:

1. (Identity of indiscernibles) $d(\Omega_0, \Omega_1) = 0 \Leftrightarrow \Omega_0 = \Omega_1$.
 - If $\Omega_0 = \Omega_1$ then $v = 0$ and $d(\Omega_0, \Omega_1) = 0$.
 - If $d(\Omega_0, \Omega_1) = 0$ that implies $\forall t \in [0, 1]$, $v(t, \cdot) = 0$ and the time space normal (3) is $v(t, \cdot) = (0, \mathbf{n}(t, \cdot))$. Then the tube is a cylinder and the domain Ω_t does not depend on time, consequently $\Omega_0 = \Omega_1$.
2. (Symmetry) $d(\Omega_0, \Omega_1) = d(\Omega_1, \Omega_0)$.
 - If we consider the backward tube $\hat{\zeta}(t) = \zeta(1-t) \in \mathcal{T}(\Omega_1, \Omega_0)$, that implies $\hat{v}(t, \cdot) = -v(1-t, \cdot)$, and consequently $d(\Omega_0, \Omega_1) = d(\Omega_1, \Omega_0)$.
3. (Triangle inequality) $d(\Omega_0, \Omega_2) \leq d(\Omega_0, \Omega_1) + d(\Omega_1, \Omega_2)$.
 - We consider three open sets domains in \mathbb{D} : $\Omega_i \forall i \in [0, 2]$. We denote by $\zeta_1 \in \mathcal{T}(\Omega_0, \Omega_1)$ the tube connecting Ω_0 to Ω_1 , and by $\zeta_2 \in \mathcal{T}(\Omega_1, \Omega_2)$ the tube connecting Ω_1 to Ω_2 . Let us consider the piecewise C^1 tube defined, through its characteristic function $\hat{\zeta}$ as follows:

$$\hat{\zeta}(t, x) = \begin{cases} \zeta_1(2t, x) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \zeta_2(2t-1, x) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad (5)$$

Consequently, the normal speed on the boundary Γ_t is given by:

$$\hat{v}(t, x) = \begin{cases} 2v_1(2t, x) & \text{if } 0 \leq t \leq \frac{1}{2} \\ 2v_2(2t-1, x) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad (6)$$

Now by construction $\hat{\zeta} \in \mathcal{T}(\Omega_0, \Omega_2)$ is a tube connecting Ω_0 to Ω_2 and we get:

$$\begin{aligned} d(\Omega_0, \Omega_2) &\leq \int_0^{\frac{1}{2}} \int_{\Gamma_t} |\hat{v}(t, x)| d\Gamma(x) dt + \int_{\frac{1}{2}}^1 \int_{\Gamma_t} |\hat{v}(t, x)| d\Gamma(x) dt \\ &\leq \int_0^{\frac{1}{2}} \int_{\Gamma_t} |2v_1(2t, x)| d\Gamma(x) dt + \int_{\frac{1}{2}}^1 \int_{\Gamma_t} |2v_2(2t-1, x)| d\Gamma(x) dt \\ &\leq \int_0^1 \int_{\Gamma_r} |v_1(r, x)| d\Gamma(x) dr + \int_0^1 \int_{\Gamma_u} |v_2(u, x)| d\Gamma(x) du \end{aligned} \quad (7)$$

and as v_1 (resp. v_2) is the infimum in the definition of $d(\Omega_0, \Omega_1)$ (resp. $d(\Omega_1, \Omega_2)$) up to $\varepsilon > 0$, then $\forall \varepsilon \in \mathbb{R}_+^*$ we get:

$$d(\Omega_0, \Omega_2) \leq d(\Omega_0, \Omega_1) + d(\Omega_1, \Omega_2) + 2\varepsilon \quad (8)$$

□

1.1 Tube Formulation using a Level Set Method

In this paper, we use a level set parameterization for the domain evolution. In this method the moving domain Ω_t is defined by the set of points in \mathbb{D} for which the level set function Φ is positive:

$$\Omega_t = \left\{ x \in \mathbb{D} \mid \Phi(t, x) > 0 \right\} \quad (9)$$

We denote by Φ_0 the level set function of the domain Ω_0 , and by Φ_1 the level set function of the domain Ω_1 :

$$\Omega_0 = \left\{ x \in \mathbb{D} \mid \Phi_0(x) > 0 \right\}, \quad \Omega_1 = \left\{ x \in \mathbb{D} \mid \Phi_1(x) > 0 \right\} \quad (10)$$

Using the level set formulation, the set of connecting tubes between the initial domain Ω_0 and the final domain Ω_1 becomes:

$$\mathcal{T}_{LS}(\Omega_0, \Omega_1) = \left\{ \begin{array}{l} \Phi(t, x) \in L^1(I, C^0(\bar{\mathbb{D}})) \\ \chi_{\Omega_t} \in C^0(\bar{I}, L^1(\mathbb{D})) \\ \Phi \text{ piecewise } C^0 \end{array} \right\}, \quad \left[\begin{array}{l} \Phi(0, x) = \Phi_0(x) \\ \Phi(1, x) = \Phi_1(x) \end{array} \right] \quad (11)$$

We consider a decomposition of the time interval I into a finite number of time intervals in which the level set function Φ is continuous. Therefore Φ is piecewise C^0 , which means that there exists an integer N and an increasing sequence: $(t_0 = 0 < t_1 < \dots < t_N = 1)$ with a decomposition of the time interval as follows: $I = \bigcup_{1 \leq k \leq N} \bar{I}_k$

where $I_k =]t_k, t_{k+1}[$, such that:

$$\forall k \in [1, N], \quad \Phi(t, \cdot) \Big|_{I_k} \in C^0(I_k) \quad (12)$$

Definition 2. The metric d defined by the equation (4) can be expressed, in term of the level set function Φ as follows:

$$d(\Omega_0, \Omega_1) = \inf_{\Phi \in \mathcal{T}_{LS}(\Omega_0, \Omega_1)} \int_0^1 \int_{\Gamma_t = \Phi^{-1}(0)} \frac{|\partial_t \Phi(t, x)|}{\|\nabla \Phi(t, x)\|} d\Gamma(x) dt \quad (13)$$

Indeed, using the level set formulation we have the relations:

$$\mathbf{n}(t, x) = \frac{-\nabla \Phi(t, x)}{\|\nabla \Phi(t, x)\|}, \quad \mathbf{V}(t, x) = -\partial_t \Phi(t, x) \frac{\nabla \Phi(t, x)}{\|\nabla \Phi(t, x)\|} \quad (14)$$

Then the normal speed of the boundary Γ_t turns into:

$$v(t, x) = \langle \mathbf{V}(t, x), \mathbf{n}(t, x) \rangle_{\mathbb{R}^n} = \frac{\partial_t \Phi(t, x)}{\|\nabla \Phi(t, x)\|} \quad (15)$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denotes the inner product in \mathbb{R}^n .

Assumption 1 *The function $d(\Omega_0, \Omega_1)$ expressed in term of the level set function Φ , is also a metric.*

1.2 Tube Formulation using the Federer Theorem

In this section, we consider the tube formulation through the level set method described previously, and we consider an approximation of the metric d using the Federer measure decomposition theorem.

Theorem 1. (Federer measure decomposition) *Let us consider a functional $F \in L^1(\mathbb{D})$, and $\forall h > 0$ the domain*

$$U_h(\Gamma) = \left\{ x \in \mathbb{D} \mid \|\Phi(t, x)\| < h \right\} \quad (16)$$

Then we have:

$$\int_{U_h(\Gamma)} F(x) dx = \int_{-h}^{+h} \left(\int_{\Phi^{-1}(z)} \frac{F(x)}{\|\nabla_x \Phi(x)\|} d\Gamma(x) \right) dz \quad (17)$$

Corollary 1. *Assuming the mapping:*

$$z \in [-h, +h] \rightarrow \int_{\Phi^{-1}(z)} \frac{F(x)}{\|\nabla_x \Phi(x)\|} d\Gamma \quad (18)$$

to be continuous, we obtain:

$$\int_{\Gamma} \frac{F(x)}{\|\nabla_x \Phi(x)\|} d\Gamma(x) = \frac{1}{2h} \int_{U_h(\Gamma)} F(x) dx + o(1), \quad h \rightarrow 0 \quad (19)$$

Definition 3. Using the Federer measure decomposition theorem and according to the previous corollary, we consider an approximation of the metric d denoted by d_h and defined as follows:

$$d_h(\Omega_0, \Omega_1) = \inf_{\Phi \in \mathcal{F}_{LS}(\Omega_0, \Omega_1)} \int_0^1 \frac{1}{2h} \int_{U_h(\Gamma_t)} |\partial_t \Phi(t, x)| dx dt \quad (20)$$

Lemma 2. *The approximation of the metric $d(\Omega_0, \Omega_1)$, denoted $d_h(\Omega_0, \Omega_1)$ is also a metric.*

Proof. We have to prove that the function $d_h(\Omega_0, \Omega_1)$ satisfies:

1. (Identity of indiscernibles) $d_h(\Omega_0, \Omega_1) = 0 \Leftrightarrow \Omega_0 = \Omega_1$.
 - If $\Omega_0 = \Omega_1$ then $\partial_t \Phi = 0$ and $d_h(\Omega_0, \Omega_1) = 0$.
 - If $d_h(\Omega_0, \Omega_1) = 0$ that implies $\forall t \in [0, 1]$, $\partial_t \Phi(t, \cdot) = 0$ in $\mathcal{D}_h = \bigcup_{0 \leq t \leq 1} \{t\} \times U_h(\Gamma_t)$ and that implies $\Phi = \Phi(x) \in \mathcal{D}_h$. Consequently,

the boundary $\Gamma_t \stackrel{def}{=} \left\{ x \in \mathbb{D} \mid \Phi(x) = 0 \right\}$ does not depend on time, and $\Omega_0 = \Omega_1$.

2. (Symmetry) $d_h(\Omega_0, \Omega_1) = d_h(\Omega_1, \Omega_0)$.
 - If we consider the backward tube $\hat{\Phi}(t, \cdot) = \Phi(1-t, \cdot) \in \mathcal{T}(\Omega_1, \Omega_0)$, that implies $\partial_t \hat{\Phi}(t, \cdot) = -\partial_t \Phi(1-t, \cdot)$, and $d_h(\Omega_0, \Omega_1) = d_h(\Omega_1, \Omega_0)$.
3. (Triangle inequality) $d_h(\Omega_0, \Omega_2) \leq d_h(\Omega_0, \Omega_1) + d_h(\Omega_1, \Omega_2)$.
 - We assume three open sets domains in $\mathbb{D} : \Omega_i \forall i \in [0, 2]$. We denote by $\Phi_1 \in \mathcal{T}_{LS}(\Omega_0, \Omega_1)$ the tube connecting Ω_0 to Ω_1 , and by $\Phi_2 \in \mathcal{T}_{LS}(\Omega_1, \Omega_2)$ the tube connecting Ω_1 to Ω_2 . Let us consider the piecewise C^1 tube defined, through its level set function Φ as follows:

$$\bar{\Phi}(t, x) = \begin{cases} \Phi_1(2t, x) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \Phi_2(2t-1, x) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad (21)$$

Consequently, the time derivative of level set function Φ on the domain $U_h(\Gamma_t)$ is given by:

$$\partial_t \bar{\Phi}(t, x)(t, x) = \begin{cases} 2 \partial_t \Phi_1(2t, x) & \text{if } 0 \leq t \leq \frac{1}{2} \\ 2 \partial_t \Phi_2(2t-1, x) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad (22)$$

Now by construction $\bar{\Phi} \in \mathcal{T}_{LS}(\Omega_0, \Omega_2)$ is a tube connecting Ω_0 to Ω_2 and we get:

$$\begin{aligned} d_h(\Omega_0, \Omega_2) &\leq \int_0^{\frac{1}{2}} \frac{1}{2h} \int_{U_h(\Gamma_t)} |\partial_t \bar{\Phi}(t, x)| dx dt + \int_{\frac{1}{2}}^1 \frac{1}{2h} \int_{U_h(\Gamma_t)} |\partial_t \bar{\Phi}(t, x)| dx dt \\ &\leq \frac{1}{2h} \left[\int_0^{\frac{1}{2}} \int_{U_h(\Gamma_t)} |2 \partial_t \Phi_1(2t, x)| dx dt + \int_{\frac{1}{2}}^1 \int_{U_h(\Gamma_t)} |2 \partial_t \Phi_2(2t-1, x)| dx dt \right] \\ &\leq \int_0^1 \frac{1}{2h} \int_{U_h(\Gamma_t)} |\partial_t \Phi_1(r, x)| dx dr + \int_0^1 \frac{1}{2h} \int_{U_h(\Gamma_t)} |\partial_t \Phi_2(u, x)| dx du \end{aligned} \quad (23)$$

and as Φ_1 (resp. Φ_2) is the infimum in the definition of $d_h(\Omega_0, \Omega_1)$ (resp. $d_h(\Omega_1, \Omega_2)$) up to $\varepsilon > 0$. Then $\forall \varepsilon \in \mathbb{R}_+^*$, we get

$$d_h(\Omega_0, \Omega_2) \leq d_h(\Omega_0, \Omega_1) + d_h(\Omega_1, \Omega_2) + 2\varepsilon \quad (24)$$

□

2 Shape Identification Problem

We address the question concerning the shape identification of a given smooth domain. A commonly used approach in shape analysis consists in choosing a level set formulation for the evolution of moving domain. The main advantage of a level set formulation concerns the easy generation of topological changes during the evolution process.

2.1 Shape Identification using a Level Set Method

Let us denote by $\Omega_* \in \mathbb{D}$ a smooth domain to identify and by χ_{Ω_*} , its characteristic function satisfying: $\chi_{\Omega_*} \in H^s(\mathbb{D})$, $0 < s < \frac{1}{2}$. Following [1], the evaluation of the distance between the given domain Ω_* and the moving domain Ω_t is made by the use of a metric associated to the Hilbert space H^s denoted $\delta_s(\Omega, \Omega_*)$ and defined by:

$$\begin{aligned} \forall s \in]0, \frac{1}{2}[, \quad \delta_s(\Omega, \Omega_*) &= \|\chi_\Omega - \chi_{\Omega_*}\|_{H^s(\mathbb{D})} \\ &= \|\chi_\Omega - \chi_{\Omega_*}\|_{L^2(\mathbb{D})} + \|\chi_\Omega - \chi_{\Omega_*}\|_s \end{aligned} \quad (25)$$

where

$$\|\chi_\Omega\|_s^2 = \int_{\mathbb{D}} \int_{\mathbb{D}} |\chi_\Omega(x) - \chi_\Omega(y)|^2 G(x, y) \, dx dy \quad (26)$$

and where the kernel function defined by: $G(x, y) = |x - y|^{-(n+2s)}$ is singular on the diagonal $\Delta = \{(x, x) \in \mathbb{D} \times \mathbb{D}, x \in \mathbb{D}\}$.

2.1.1 Shape Analysis via the Speed Method

Finally, we use the concept of speed method from shape analysis [3] to compute the shape derivative of the metric $\delta_s(\Omega, \Omega_*)$ which corresponds to a gradient direction for the underlying shape optimization problem:

$$\min_{\Omega \in \mathbb{D}} \delta_s(\Omega, \Omega_*) \quad (27)$$

Definition 4. Let us consider an open set domain Ω where $\Gamma = \partial\Omega$ is of class C^1 . We define the eulerian derivative of the functional J in the direction of a perturbation vector field $\mathbf{W} \in C_0^1(\mathbb{D}; \mathbb{D})$ by

$$dJ(\Gamma, \mathbf{W}) = \left. \frac{\partial J(\Gamma(t + \varepsilon))}{\partial \varepsilon} \right|_{\varepsilon=0} \quad (28)$$

Lemma 3. *The functional $\delta_s(\Omega, \Omega_*)$ is shape derivative for perturbation vector fields $\mathbf{V} \in C_0^1(\mathbb{D}, \mathbb{D})$, and expressed as follows:*

$$d\delta_s((\Omega, \Omega_*); \mathbf{V}) = \int_{\Gamma} F(x) \langle \mathbf{V}(0, x), \mathbf{n}(x) \rangle_{\mathbb{R}^n} d\Gamma(x) \quad (29)$$

where $d\Gamma$ is the arclength measure on Γ and where:

$$F(x) = \frac{1 - 2\chi_{\Omega_*}(x)}{2\|\chi_{\Omega} - \chi_{\Omega_*}\|_{L^2(\mathbb{D})}} + \frac{\int_{\mathbb{D}} [1 - 2\chi_{\Omega_t}(y) + 2[\chi_{\Omega_*}(y) - \chi_{\Omega_*}(x)]] G(x, y) dy}{\|\chi_{\Omega} - \chi_{\Omega_*}\|_s} \quad (30)$$

Proof. See [1]. □

2.2 Shape Identification using a Galerkin-Level Set Strategy

Generally, the parameterization of the level set function Φ is done by the *oriented distance function* denoted b_{Ω_t} , see [2, 3] for references:

$$\Phi(t, x) = -b_{\Omega_t}(x) \quad (31)$$

where $b_{\Omega}(x)$ is also called *signed distance function* and is defined as follows:

$$b_{\Omega}(x) = d_{\Omega}(x) - d_{\mathbb{C}\Omega}(x) \quad \text{with} \quad d_A(x) = \inf_{y \in A} |y - x| \quad (32)$$

The choice of the *oriented distance function* for the parameterization of the level set function can be necessary for having a high accuracy of the boundary approximation. However, the choice of the *oriented distance function* implies an expansive computational cost owing to the complexity of its evaluation and imposes a reinitialization during the evolution process. Consequently, according to the fact that in this paper we focus on topological changes without considering the approximation of the boundary as an essential point, we use a new approach called Galerkin-Level Set method.

2.2.1 Galerkin-Level Set Strategy

The Galerkin-Level Set strategy consists in parameterizing the level set function in a finite dimensional subspace \mathcal{E} , spanned by linear independent functions defined over \mathbb{D} : $\mathcal{E} = \{E_1, \dots, E_m\}$. We denote by $\Lambda(t) = (\lambda_1(t), \dots, \lambda_m(t))$ the parameter vector of the Galerkin decomposition of Φ in the basis \mathcal{E} :

$$\Phi(t, x) = \sum_{k=1}^m \lambda_k(t) E_k(x) \quad (33)$$

Consequently, using the Galerkin decomposition of the level set function, the parameterization of the moving domain $\Omega(t)$ is defined as follows:

$$\Omega(t) = \left\{ x \in \mathbb{D} \mid \Phi(t, x) = \sum_{k=1}^m \lambda_k(t) E_k(x) > 0 \right\} \quad (34)$$

2.2.2 Level Set Equation

In a level set formulation, the moving domain evolves by advecting the level set function Φ following the flow of the shape gradient. Then, in a traditional level set formulation, the transport equation is a Partial Differential Equation (PDE) of Hamilton-Jacobi type:

$$\begin{cases} \partial_t \Phi(t, x) + \rho F(x) \|\nabla \Phi(t, x)\| = 0, & \rho > 0 \\ \Phi(0, x) = \Phi_0(x), & (t, x) \in [0, \tau] \times \Omega_t \end{cases} \quad (35)$$

Remark 1 *The main advantage of the Galerkin-Level Set method compared to the traditional level set formulation concerns the level set equation that turns, in the Galerkin-Level Set method, into a system of ordinary differential equations.*

Lemma 4. *Using the Galerkin-Level Set strategy (34), the level set equation turns into a system of m ordinary differential equations:*

$$\begin{cases} \partial_t \Lambda(t) + \rho \mathcal{F}(t, x) = 0, & \rho > 0 \\ \Lambda(0) = \Lambda_0, & (t, x) \in [0, \tau] \times \Omega_t \end{cases} \quad (36)$$

where

$$\mathcal{F}(t, x) = \left(\int_{\Gamma} \frac{F(x)}{\|\nabla \Phi(t, x)\|} E_1(x) d\Gamma(x), \dots, \int_{\Gamma} \frac{F(x)}{\|\nabla \Phi(t, x)\|} E_m(x) d\Gamma(x) \right) \quad (37)$$

Proof. According to the Galerkin-Level Set strategy, consisting in the decomposition of function Φ (33), the shape derivative of the functional $\delta_s(\Omega, \Omega_*)$ with respect to the vector of parameters $\Lambda(t)$ turns into:

$$d \delta_s((\Omega, \Omega_*); \mathbf{V}) = \sum_{k=1}^m \partial_t \lambda_k(t) \int_{\Gamma} F(x) \frac{E_k(x)}{\|\nabla \Phi(t, x)\|} d\Gamma(x) \quad (38)$$

where only the vector of parameters $\Lambda(t)$ depends on time. A sufficient condition to decrease the shape gradient is to choose:

$$\forall k \in [1, m], \forall \rho \in \mathbb{R}_+^*, \quad \partial_t \lambda_k(t) = -\rho \int_{\Gamma} F(x) \frac{E_k(x)}{\|\nabla \Phi(t, x)\|} d\Gamma(x) \quad (39)$$

Finally, considering the level set equation we obtain a system of m ordinary differential equations (36). \square

Corollary 2. *Substituting the approximation of the boundary integral calculus from the equation (19), into the system of m ordinary differential equations (37), we obtain an approximation of the vector \mathcal{F} defined as follows:*

$$\tilde{\mathcal{F}}(t, x) = \left(\frac{1}{2h} \int_{U_h(\Gamma)} F(x) E_1(x) dx, \dots, \frac{1}{2h} \int_{U_h(\Gamma)} F(x) E_m(x) dx \right) \quad (40)$$

Note that in this new formulation the main advantage is that the denominator term $\|\nabla \Phi(x)\|$ has been eliminated.

From now, we use the previous corollary for the level set equation and we consider the following algorithm.

Algorithm 1:

1. **Initialization:** Choose an initial vector of parameters $\Lambda_0 = (\lambda_1^0, \dots, \lambda_m^0)$. Initialize the level set function $\Phi_0(x) = \sum_{l=1}^m \lambda_l^0 E_l(x)$. Set $k = 0$.
2. **Shape gradient direction:** Find the tubular neighborhood $U_h(\Gamma_{t_k})$ of the zero level set Γ_{t_k} of the actual level set function $\Phi(t_k, x)$. Compute $\tilde{\mathcal{F}}(t_k, x)$ from the equation (40).
3. **Update:** Perform a time step in the level set equation (36) to update $\Lambda(t_k)$. Let $\Lambda(t_{k+1})$ denote this update: $\Lambda(t_{k+1}) = \Lambda(t_k) - \rho \tilde{\mathcal{F}}(t_k, x)$, $\rho > 0$. Update the function $\Phi(t_{k+1}, x) = \sum_{l=1}^m \lambda_l(t_{k+1}) E_l(x)$. Set $k = k + 1$ and go to 2.

2.3 Numerical Experiment

We present a numerical experiment based on the algorithm 1 for a 3D shape identification problem using the Galerkin-Level Set method described in the previous section. In this numerical experiment, the given domain Ω_* to identify is the gray matter of a human brain. We consider a Galerkin-Level Set expansion of the level set function Φ in Fourier series of dimension $m = 25^3$; note that in this 3D case the level set function Φ is in \mathbb{R}^4 . We start with a smooth initial domain $\Omega_{t=0}$ corresponding to the lower frequency of the Fourier series: see left-hand picture in Fig. 2. The algorithm detects the contour of the human brain after only 8 iterations.

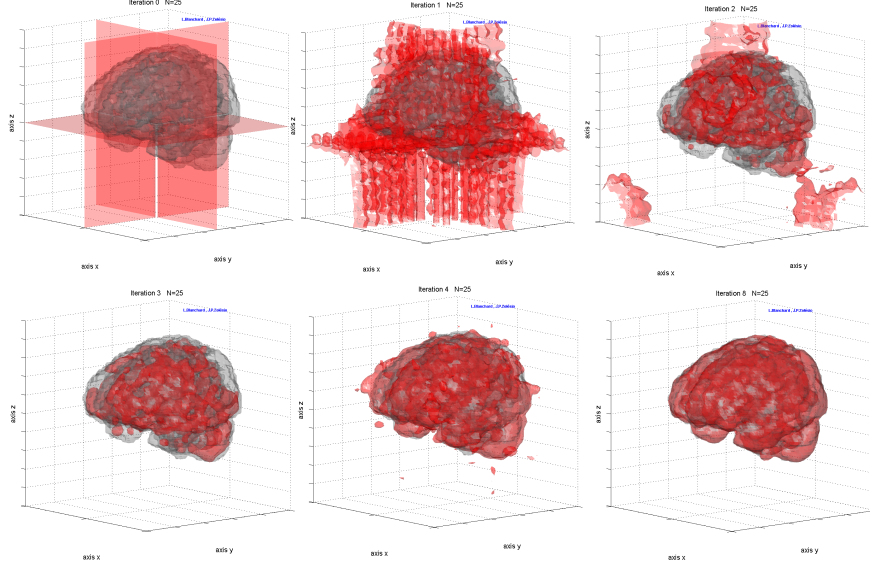


Fig. 2 Shape identification of gray matter of human brain using a Galerkin-Level Set method

3 Geodesic Tube Formulation using Moving Domain

3.1 Tube Formulation using a Galerkin Strategy

The tube path between Ω_0 and Ω_1 is made by a Galerkin-Level Set approach. The moving domain Ω_t of the tube evolution defined by the equation (1) is parameterized by the Galerkin-Level Set formulation and defined as follows:

$$\Omega(t) = \left\{ x \in \mathbb{D} \mid \Phi(t, x) = \sum_{k=1}^m \lambda_k(t) E_k(x) > 0 \right\} \quad (41)$$

where $\Lambda(t) = (\lambda_1(t), \dots, \lambda_m(t)) \in \mathbb{R}^m$ is the vector of parameters in the Galerkin expansion of the level set function. The first step consists in identifying the initial domain Ω_0 and the final domain Ω_1 through the research of the parameters $\Lambda_0 = (\lambda_1^0, \dots, \lambda_m^0) \in \mathbb{R}^m$ and $\Lambda_1 = (\lambda_1^1, \dots, \lambda_m^1) \in \mathbb{R}^m$ which satisfy the equations:

$$\Phi_0(x) = \sum_{k=1}^m \lambda_k^0 E_k(x), \quad \Phi_1(x) = \sum_{k=1}^m \lambda_k^1 E_k(x) \quad (42)$$

Thus, the feasible set of connecting tubes between Ω_0 and Ω_1 through the Galerkin-Level Set formulation turns into:

$$\mathcal{T}_\Lambda(\Omega_0, \Omega_1) = \left\{ \Lambda(t) \in (L^2(I))^m, \left[\begin{array}{l} \Lambda(0) = \Lambda_0 \\ \Lambda(1) = \Lambda_1 \end{array} \right] \right\} \quad (43)$$

Remark 2 *The feasible set of connecting tubes between Ω_0 and Ω_1 is not empty. Indeed, if we consider the vector of parameters $\Lambda(t)$ as a convex combination of Λ_0 and Λ_1 : $\Lambda(t) = \Lambda_1 t + \Lambda_0(1-t)$, we have $\Lambda(t) \in \mathcal{T}_\Lambda(\Omega_0, \Omega_1)$. Moreover, the parameters $\Lambda(t)$ defined as a convex combination of Λ_0 and Λ_1 generate an admissible tube that we use for the initialization during the tube optimization process.*

3.2 Geodesic Tube Construction between Two Domains

We focus on the construction of an optimal tube connecting the initial domain Ω_0 to the final domain Ω_1 , this optimal tube is also called a geodesic tube. The question is to determine, through the use of shape metrics $d(\Omega_0, \Omega_1)$ and $d_h(\Omega_0, \Omega_1)$, which tube is an optimal tube among all those tubes in the admissible set (see Fig. 3).

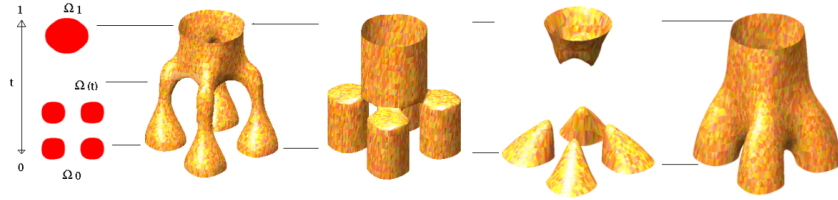


Fig. 3 Different continuous tubes between Ω_0 and Ω_1

Let us consider the metrics d and d_h defined by (4) and (20) that we can rewrite as follows:

$$\begin{aligned} d(\Omega_0, \Omega_1) &= \inf_{\Phi \in \mathcal{T}_{LS}(\Omega_0, \Omega_1)} \int_0^1 J(\Gamma_t) dt \\ d_h(\Omega_0, \Omega_1) &= \inf_{\Phi \in \mathcal{T}_{LS}(\Omega_0, \Omega_1)} \int_0^1 J_h(\Gamma_t) dt \end{aligned} \quad (44)$$

where the functionals $J(\Gamma_t)$ and $J_h(\Gamma_t)$ are defined by:

$$\begin{aligned} J(\Gamma_t) &= \int_{\Gamma_t} |v(t, x)| d\Gamma(x) = \int_{\Gamma_t} \frac{|\partial_t \Phi(t, x)|}{\|\nabla \Phi(t, x)\|} d\Gamma(x) \\ J_h(\Gamma_t) &= \frac{1}{2h} \int_{U_h(\Gamma_t)} |\partial_t \Phi(t, x)| dx \end{aligned} \quad (45)$$

Then, in order to solve the problem concerning the geodesic tube, that is to say to compute the metrics d or d_h defined by (44), we use a gradient method based on the computation of the shape derivative.

Lemma 5. *According to (28), the eulerian derivative of the functional J in the direction of a perturbation vector field $\mathbf{W} \in C_0^1(\mathbb{D}; \mathbb{D})$ is:*

$$\begin{aligned} dJ(\Gamma_t, \mathbf{W}) &= \int_{\Gamma_t} \partial_\varepsilon |v(t + \varepsilon, x)| \Big|_{\varepsilon=0} d\Gamma(x) \\ &+ \int_{\Gamma_t} \left[\frac{\partial |v(t, x)|}{\partial n} + H(t, x) |v(t, x)| \right] \langle \mathbf{W}(t, x), \mathbf{n}(t, x) \rangle_{\mathbb{R}^n} d\Gamma(x) \end{aligned} \quad (46)$$

where H is the mean curvature. Using the level set formulation the eulerian derivative of the functional J turns into:

$$\begin{aligned} dJ(\Gamma_t, \mathbf{W}) &= \int_{\Gamma_t} \left[\frac{\text{sign}(\partial_t \Phi)}{\|\nabla \Phi\|} \partial_\varepsilon (\partial_t \Phi) \Big|_{\varepsilon=0} - |\partial_t \Phi| \frac{1}{\|\nabla \Phi\|^3} \nabla \Phi \cdot \nabla (\partial_\varepsilon \Phi) \Big|_{\varepsilon=0} \right. \\ &+ \left(-\text{sign}(\partial_t \Phi) \frac{\nabla \Phi}{\|\nabla \Phi\|} \cdot \frac{\nabla (\partial_t \Phi)}{\|\nabla \Phi\|} + 2|\partial_t \Phi| \frac{\nabla \Phi}{\|\nabla \Phi\|} \cdot \left[\frac{D^2 \Phi}{\|\nabla \Phi\|^2} \cdot \frac{\nabla \Phi}{\|\nabla \Phi\|} \right] - \right. \\ &\left. - \frac{|\partial_t \Phi|}{\|\nabla \Phi\|^2} \Delta \Phi \right) \frac{\partial_\varepsilon \Phi}{\|\nabla \Phi\|} \Big|_{\varepsilon=0} d\Gamma(x) \end{aligned} \quad (47)$$

Proof. According to the equation (46), the eulerian derivative of the functional J_{ts} in the direction of a perturbation vector field \mathbf{W} turns into:

$$\begin{aligned} \partial_\varepsilon |v(t + \varepsilon, x)| \Big|_{\varepsilon=0} &= \partial_\varepsilon \left(\frac{|\partial_t \Phi|}{\|\nabla \Phi\|} \right) \Big|_{\varepsilon=0} \\ &= \frac{\text{sign}(\partial_t \Phi)}{\|\nabla \Phi\|} \partial_\varepsilon (\partial_t \Phi) \Big|_{\varepsilon=0} + \partial_\varepsilon \left(\frac{1}{\|\nabla \Phi\|} \right) \Big|_{\varepsilon=0} |\partial_t \Phi| \\ &= \frac{\text{sign}(\partial_t \Phi)}{\|\nabla \Phi\|} \partial_\varepsilon (\partial_t \Phi) \Big|_{\varepsilon=0} - |\partial_t \Phi| \frac{1}{\|\nabla \Phi\|^3} \nabla \Phi \cdot \nabla (\partial_\varepsilon \Phi) \Big|_{\varepsilon=0} \end{aligned} \quad (48)$$

and

$$\frac{\partial |v(t, x)|}{\partial n} + H |v(t, x)| = \frac{-\nabla \Phi}{\|\nabla \Phi\|} \cdot \nabla \left(\frac{|\partial_t \Phi|}{\|\nabla \Phi\|} \right) + \nabla \cdot \left(\frac{-\nabla \Phi}{\|\nabla \Phi\|} \right) \frac{|\partial_t \Phi|}{\|\nabla \Phi\|} \quad (49)$$

$$\begin{aligned}
& \frac{-\nabla\Phi}{\|\nabla\Phi\|} \cdot \nabla\left(\frac{|\partial_t\Phi|}{\|\nabla\Phi\|}\right) + \left[-\nabla\left(\frac{1}{\|\nabla\Phi\|}\right) \cdot \nabla\Phi - \frac{\Delta\Phi}{\|\nabla\Phi\|}\right] \frac{|\partial_t\Phi|}{\|\nabla\Phi\|} \\
&= \frac{-\nabla\Phi}{\|\nabla\Phi\|} \cdot \frac{\nabla(|\partial_t\Phi|)}{\|\nabla\Phi\|} - 2|\partial_t\Phi| \frac{\nabla\Phi}{\|\nabla\Phi\|} \cdot \nabla\left(\frac{1}{\|\nabla\Phi\|}\right) - \frac{|\partial_t\Phi|}{\|\nabla\Phi\|^2} \Delta\Phi \\
&= -\text{sign}(\partial_t\Phi) \frac{\nabla\Phi}{\|\nabla\Phi\|} \cdot \frac{\nabla(|\partial_t\Phi|)}{\|\nabla\Phi\|} + \\
&+ 2|\partial_t\Phi| \frac{\nabla\Phi}{\|\nabla\Phi\|} \cdot \left[\frac{D^2\Phi}{\|\nabla\Phi\|^2} \cdot \frac{\nabla\Phi}{\|\nabla\Phi\|}\right] - \frac{|\partial_t\Phi|}{\|\nabla\Phi\|^2} \Delta\Phi
\end{aligned} \tag{50}$$

□

Lemma 6. According to (28), the eulerian derivative of the functional J_h in the direction of a perturbation vector field $\mathbf{W} \in C_0^1(\mathbb{D}; \mathbb{D})$ is:

$$\begin{aligned}
dJ_h(\Gamma, \mathbf{W}) &= \frac{1}{2h} \int_{\mathbb{D}} \left[\partial_\varepsilon(|\partial_t\Phi(t+\varepsilon, x)|) \Big|_{\varepsilon=0} \rho_h \circ b_{\Omega_t}(x) \right] dx \\
&+ \frac{1}{2h} \int_{\mathbb{D}} \left[|\partial_t\Phi(t, x)| \partial_\varepsilon(\rho_h \circ b_{\Omega_{t+\varepsilon}}(x)) \Big|_{\varepsilon=0} \right] dx
\end{aligned} \tag{51}$$

where the function ρ_h is defined by: $\rho_h(x) = \begin{cases} \frac{x}{h} + 1 & \text{if } x \in [-h, 0] \\ \frac{-x}{h} + 1 & \text{if } x \in [0, h] \\ 0 & \text{if } x \in \mathbb{R} \setminus [-h, h] \end{cases}$.

Proof. Due to the fact that $\rho_h \circ b_{\Omega_t}(x) \Big|_{\Gamma} = 1$, and using the fact that $\text{supp}(\rho_h \circ b_{\Omega_t}) \subseteq U_h(\Gamma)$ we can rewrite the functional J_h as follows

$$J_h(\Gamma) = \frac{1}{2h} \int_{\mathbb{D}} |\partial_t\Phi(t, x)| \rho_h \circ b_{\Omega_t}(x) dx \tag{52}$$

Consequently, the eulerian derivative of the functional J_h turns into the equation (51) where:

$$\partial_\varepsilon(|\partial_t\Phi(t+\varepsilon, x)|) \Big|_{\varepsilon=0} = \text{sign}(\partial_t\Phi) \partial_\varepsilon(|\partial_t\Phi(t, x)|) \Big|_{\varepsilon=0}. \tag{53}$$

Using $\partial_\varepsilon(b_{\Omega_t}(x)) + \nabla b_{\Omega_t}(x) \cdot \mathbf{W} \circ p = 0$ and $\nabla b_{\Omega_t}(x) = \mathbf{n}(t, x)$, we get

$$\begin{aligned}
\partial_\varepsilon(\rho_h \circ b_{\Omega_t}(x)) \Big|_{\varepsilon=0} &= \rho_h' \circ b_{\Omega_t}(x) \partial_\varepsilon(b_{\Omega_t}(x)) \\
&= -\rho_h' \circ b_{\Omega_t}(x) \langle \mathbf{W}, \mathbf{n} \rangle_{\mathbb{R}^n} \\
&= -\rho_h' \circ b_{\Omega_t}(x) \frac{\partial_\varepsilon\Phi}{\|\nabla\Phi\|}
\end{aligned} \tag{54}$$

The derivative of the function $\rho_h(x)$ is defined by:

$$\rho'_h(x) = \begin{cases} \frac{1}{h} & \text{if } x \in [-h, 0] \\ -\frac{1}{h} & \text{if } x \in [0, h] \\ 0 & \text{if } x \in \mathbb{R} \setminus [-h, h] \end{cases} = \begin{cases} \frac{1}{h}(1 - 2\chi_{\Omega_t}(x)) & \text{if } x \in U_h(\Gamma_t) \\ 0 & \text{if } x \in \mathbb{R} \setminus [-h, h] \end{cases}$$

Finally, we get for the eulerian derivative of the functional J_h in the direction of a perturbation vector field $\mathbf{W} \in C_0^1(\mathbb{D}; \mathbb{D})$:

$$\begin{aligned} dJ_h(\Gamma_t, \mathbf{W}) &= \frac{1}{2h} \int_{\mathbb{D}} \left[\text{sign}(\partial_t \Phi) \partial_\varepsilon (\partial_t \Phi(t, x)) \Big|_{\varepsilon=0} \rho_h \circ b_{\Omega_t}(x) \right] dx \\ &\quad - \frac{1}{2h} \int_{\mathbb{D}} \left[|\partial_t \Phi(t, x)| \rho'_h \circ b_{\Omega_t}(x) \frac{\partial_\varepsilon \Phi}{\|\nabla \Phi\|} \right] dx \end{aligned} \quad (55)$$

□

3.2.1 Polynomial Decomposition of the Parameter $\Lambda(t)$

We continue the study of a geodesic tube through a tube formulation using a Galerkin-Level set strategy. Consequently, $\Lambda(t) \in \mathcal{T}_\Lambda(\Omega_0, \Omega_1)$ represents the parameters of the optimization process. For complexity reason, we consider a polynomial decomposition of the parameter $\Lambda(t)$ as follows:

$$\Lambda(t) = P_\alpha(t) \Lambda_1 + (1 - P_\alpha(t)) \Lambda_0, \quad P_\alpha(t) = \sum_{i=1}^M \alpha_i e_i(t) \quad (56)$$

where $\alpha = (\alpha_1, \dots, \alpha_M)$ are the coefficients of the decomposition of the polynomial $P_\alpha(t)$ in the basis $\{e_1(t), \dots, e_M(t)\}$. Consequently, the feasible set of connecting tubes defined by (43) with initial and final conditions on $\Lambda(t)$ turns into a feasible set with initial and final conditions on the polynomial P_α defined as follows:

$$\mathcal{T}_\alpha(\Omega_0, \Omega_1) = \left\{ \alpha \in \mathbb{R}^M, \begin{cases} P_\alpha(0) = 0 \\ P_\alpha(1) = 1 \end{cases} \right\} \quad (57)$$

Let us consider the metrics d and d_h defined by (4) and (20) that we can rewrite as follows:

$$\begin{aligned} d(\Omega_0, \Omega_1) &= \inf_{\alpha \in \mathcal{T}_\alpha(\Omega_0, \Omega_1)} \int_0^1 \tilde{J}(\Gamma_t) dt \\ d_h(\Omega_0, \Omega_1) &= \inf_{\alpha \in \mathcal{T}_\alpha(\Omega_0, \Omega_1)} \int_0^1 \tilde{J}_h(\Gamma_t) dt \end{aligned} \quad (58)$$

where the functionals $\tilde{J}(\Gamma_t)$ and $\tilde{J}_h(\Gamma_t)$ are defined by:

$$\begin{aligned}\tilde{J}(\Gamma_t) &= |\dot{P}_\alpha(t)| \int_{\Gamma_t} \frac{|\Phi_1(x) - \Phi_0(x)|}{\|\nabla\Phi\|} d\Gamma(x) \\ \tilde{J}_h(\Gamma_t) &= \frac{|\dot{P}_\alpha(t)|}{2h} \int_{U_h(\Gamma_t)} |\Phi_1(x) - \Phi_0(x)| dx\end{aligned}\quad (59)$$

Then, in order to solve the problem concerning the geodesic tube, that is to say to compute the metrics d or d_h defined by (58), we use a gradient method based on the computation of the shape derivative.

Assumption 2 *The shape derivative of the functional J defined by (46) can be rewritten as follows:*

$$dJ(\Gamma_t, \mathbf{W}) = \left. \frac{\partial J(\alpha + \varepsilon h)}{\partial \varepsilon} \right|_{\varepsilon=0} = \langle h, \nabla J(\Gamma_t) \rangle_{\mathbb{R}^M} \quad (60)$$

where $\forall i \in [1, M]$:

$$\begin{aligned}(\nabla J(\Gamma_t))_i &= \dot{e}_i(t) \operatorname{sign}(\dot{P}_\alpha(t)) \int_{\Gamma_t} \frac{|\Phi_1(x) - \Phi_0(x)|}{\|\nabla\Phi\|} d\Gamma(x) \\ &+ e_i(t) |\dot{P}_\alpha(t)|, \int_{\Gamma_t} \frac{|\Phi_1(x) - \Phi_0(x)|}{\|\nabla\Phi\|} K(t, x) d\Gamma(x)\end{aligned}\quad (61)$$

and

$$\begin{aligned}K(t, x) &= \left[-2 \frac{\nabla\Phi \cdot (\nabla\Phi_1(x) - \nabla\Phi_0(x))}{\|\nabla\Phi\|^2} + \right. \\ &+ 2(\Phi_1(x) - \Phi_0(x)) \frac{\nabla\Phi}{\|\nabla\Phi\|} \cdot \left[\frac{D^2\Phi}{\|\nabla\Phi\|^2} \cdot \frac{\nabla\Phi}{\|\nabla\Phi\|} \right] \\ &\left. - (\Phi_1(x) - \Phi_0(x)) \frac{\Delta\Phi}{\|\nabla\Phi\|^2} \right]\end{aligned}\quad (62)$$

Assumption 3 *The shape derivative of the functional J_h defined by (51) can be rewritten as follows:*

$$dJ_h(\Gamma_t, \mathbf{W}) = \left. \frac{\partial J_h(\alpha + \varepsilon h)}{\partial \varepsilon} \right|_{\varepsilon=0} = \langle h, \nabla J_h(\Gamma_t) \rangle_{\mathbb{R}^M} \quad (63)$$

where $\forall i \in [1, M]$:

$$\begin{aligned}
(\nabla J_h(\Gamma_t))_i &= \frac{\dot{e}_i(t)}{2h} \text{sign}(\dot{P}_\alpha(t)) \int_{U_h(\Gamma_t)} |\Phi_1(x) - \Phi_0(x)| dx \\
&\quad - \frac{e_i(t)}{2h^2} |\dot{P}_\alpha(t)| \int_{U_h(\Gamma_t)} (1 - 2\chi_{\Omega_t}(x)) |\Phi_1(x) - \Phi_0(x)| \\
&\quad \frac{(\Phi_1(x) - \Phi_0(x))}{\|\nabla \Phi\|} dx
\end{aligned} \tag{64}$$

Algorithm 2:

1. **Initialization:** Choose an initial vector of parameters $\Lambda(t)$ defined by (56) which generate an admissible connecting tube between Ω_0 and Ω_1 through the choice of the parameter α . Initialize the level set function $\Phi(t, x) = P_\alpha(t) \Phi_1(x) + (1 - P_\alpha(t)) \Phi_0(x)$. Set $k = 0$.
2. **Shape gradient direction:** For every $t \in I$, find the tubular neighborhood $U_h(\Gamma_t)$ of the zero level set Γ_t of the actual level set function $\Phi(t, x)$. Compute $\nabla J_h(\Gamma_t)$ from the equation (64).
3. **Update:**
 - Perform a time step to update α .
 - Let α^+ denote this update: $\alpha^+ = \alpha - \rho \int_0^1 \nabla J_h(\Gamma_t) dt$, $\rho > 0$.
 - Update the function $\Phi^+(t, x) = P_{\alpha^+}(t) \Phi_1(x) + (1 - P_{\alpha^+}(t)) \Phi_0(x)$.
 - Set $k = k + 1$ and go to (2).

3.3 Numerical Experiment of a Geodesic Tube Construction

We present a numerical experiment based on the algorithm 2 for a 3D tube optimization. Fig. 4 shows tubes obtained during the optimization process for different iterations. From Fig. 5 we can see that the tube obtained after seven iterations has a more homogeneous distribution of the functional values $J_h(\Gamma_t)$ compared to the initial tube. The result of this optimization process is the construction of a smoother tube than the initial tube.

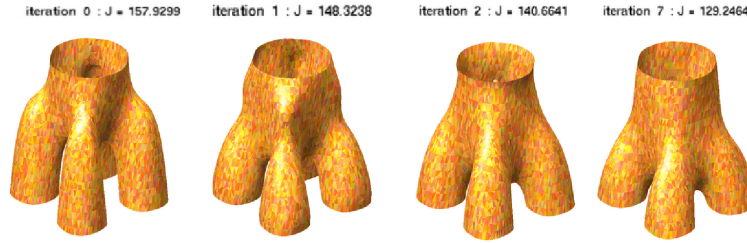


Fig. 4 Tube optimization using the metric $d_h(\Omega_0, \Omega_1)$

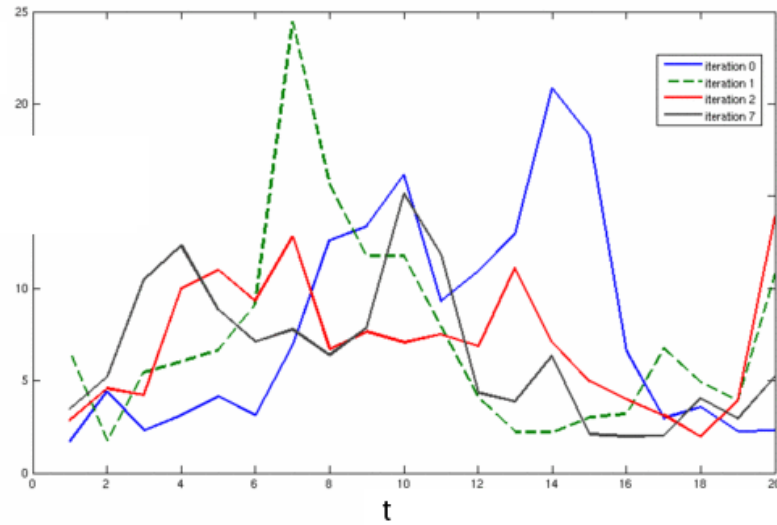


Fig. 5 Distribution of the functional values $J_h(I_t)$ for tube obtained during the optimization process of Fig. 4

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