Exponential stability of the system of transmission of the wave equation with a delay term in the boundary feedback

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Abstract. We consider a system of transmission of the wave equation with Neumann feedback control that contains a delay term and that acts on the exterior boundary. First, we prove under some assumptions that the closed-loop system generates a C_0-semigroup of contractions on an appropriate Hilbert space. Then, under further assumptions, we show that the closed-loop system is exponentially stable. To establish this result, we introduce a suitable energy function and use multiplier method together with an estimate taken from [3] (Lemma 7.2) and compactness-uniqueness arguments.

Keywords: Wave equation, transmission problem, time delays, boundary stabilization, exponential stability.

1 Introduction

It is by now well-known that certain infinite-dimensional second-order systems are not robust with respect to arbitarily small delays in the damping. This lack of stability robustness was first shown to hold for the one-dimensional wave equation ([2]). Later, further examples illustrating this phenomenon were considered in [1]: the two-dimensional wave equation with damping introduced through Neumann-type boundary conditions on one edge of a square boundary and the Euler-Bernoulli beam equation in one dimension with damping introduced through a specific set of boundary conditions on the right end point.

Recently, Xu et al [9] established sufficient conditions that guarantee the exponential stability of the one-dimensional wave equation with a delay term in the boundary feedback. Nicaise and Pignotti [5] extended this result to the multi-dimensional wave equation with a delay term in the boundary or internal feedbacks. The same type of result was obtained by Nicaise and Rebiai [6] for the Schrödinger equation.

Motivated by the references [9], [5] and [6]; we investigate in this paper the problem of exponential stability for the system of transmission of the wave equation with a delay term in the boundary feedback.

Let Ω be an open bounded domain of $\mathbb{R}^n$ with a boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$. Let Γ_0

with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \overline{\Gamma_0} \cap \overline{\Gamma_2} = \emptyset$ be a regular hypersurface of class C^2 which separates Ω into two domains Ω_1 and Ω_2 such that $\Gamma_1 \subset \partial\Omega_1$ and $\Gamma_2 \subset \partial\Omega_2$. Furthermore, we assume that there exists a real vector field $h \in (C^2(\overline{\Omega}))^n$ such that:
$(H.1)$ The Jacobian matrix J of h satisfies

$$\int_{\Omega} J(x)\zeta(x).\zeta(x)d\Omega \geq \alpha \int_{\Omega} |\zeta(x)|^2 \, d\Omega,$$

for some constant $\alpha > 0$ and for all $\zeta \in L^2(\Omega; \mathbb{R}^n)$;
$(H.2)$ $h(x).\nu(x) \leq 0$ on Γ_1;

$(H.3)$ $h(x).\nu(x) \geq 0$ on Γ_0.
where ν is the unit normal on Γ or Γ_0 pointing towrds the exterior of Ω or Ω_1.

Let $a_1, a_2 > 0$ be given. Consider the system of transmission of the wave equation with a delay term in the boundary conditions:

$$y''(x,t) - a(x)\Delta y(x,t) = 0 \qquad \text{in } \Omega \times (0,+\infty), \tag{1}$$
$$y(x,0) = y^0(x), y'(x,0) = y^1(x,0) \qquad \text{in } \Omega, \tag{2}$$
$$y_1(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{3}$$
$$\frac{\partial y_2(x,t)}{\partial \nu} = -\mu_1 y_2'(x,t) - \mu_2 y_2'(x,t-\tau) \quad \text{on } \Gamma_2 \times (0,+\infty), \tag{4}$$
$$y_1(x,t) = y_2(x,t), \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{5}$$
$$a_1 \frac{\partial y_1(x,t)}{\partial \nu} = a_2 \frac{\partial y_2(x,t)}{\partial \nu} \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{6}$$
$$y_2'(x,t-\tau) = f_0(x,t-\tau) \qquad \text{on } \Gamma_2 \times (0,\tau). \tag{7}$$

where:

- $a(x) = \begin{cases} a_1, & x \in \Omega_1 \\ a_2, & x \in \Omega_2 \end{cases}$
- $y(x,t) = \begin{cases} y_1(x,t), (x,t) \in \Omega_1 \times (0,+\infty) \\ y_1(x,t), (x,t) \in \Omega_2 \times (0,+\infty) \end{cases}$
- $\frac{\partial .}{\partial \nu}$ is the normal derivative.
- μ_1 and μ_2 are positive real numbers.
- τ is the time delay
- y^0, y^1, f_0 are the initial data which belong to suitable spaces.

In the absence of delay, that is $\mu_2 = 0$, Liu and Williams [4] have shown that the solution of (1)-(6) decays exponentially to zero in the energy space $H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)$ provided that

$$a_1 > a_2 \tag{8}$$

and $\{\Omega, \Gamma_0, \Gamma_1, \Gamma_2\}$ satisfies $(H.1), (H.2), (H.3)$, and
$(H.4)$ $h(x).\nu(x) \geq \gamma > 0$.

The purpose of this paper is to investigate the stability of problem $(1)-(7)$ in the case where both μ_1 and μ_2 are different from zero. To this end, assume as in [5] that

$$\mu_1 > \mu_2. \tag{9}$$

and define the energy of a solution of $(1)-(7)$ by

$$E(t) = \frac{1}{2}\int_{\Omega}\left[\left|y'(x,t)\right|^2 + a(x)\left|\nabla(y(x,t)\right|^2\right]\,dx + \frac{\xi}{2}\int_{\Gamma_2}\int_0^1 \left|y(x,t-\tau\rho)\right|^2 d\rho\, d\sigma(x), \tag{10}$$

where

$$a_2\tau\mu_2 < \xi < a_2\tau(2\mu_1-\mu_2), \tag{11}$$

We show that if $\{\Omega,\Gamma_0,\Gamma_1,\Gamma_2\}$ satisfies $(H.1),(H.2)$ and $(H.3)$, then there is an exponential decay rate for $E(t)$.The proof of this result combines multipliers technique and compactness-uniqueness arguments.
The main result of this paper can be stated as follows.

Theorem 1. *Assume* $(H1)$, $(H.2),(H.3)$, (8) *and* (9). *Then there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$E(t) \leq M e^{-\omega t} E(0).$$

Theorem 1 is proved in Section 3. In Section 2, we investigate the well-posedness of system $(1)-(7)$ using semigroup theory.

2 Well-poseness of problem $(1)-(7)$

Inspired from [5] and [6], we introduce the auxilliary variable $z(x,\rho,t) = y(x,t-\tau\rho)$. With this new unknown, problem $(1)-(7)$ is equivalent to

$$\begin{aligned}
&y''(x,t) - a(x)\Delta y(x,t) = 0 && \text{in } \Omega\times(0,+\infty), && (12)\\
&y(x,0) = y^0(x), y'(x,0) = y^1(x) && \text{in } \Omega, && (13)\\
&y(x,t) = 0 && \text{on } \Gamma_1\times(0,+\infty), && (14)\\
&\frac{\partial z(x,\rho,t)}{\partial t} + \frac{1}{\tau}\frac{\partial z(x,\rho,t)}{\partial \rho} = 0 && \text{on } \Gamma_2\times(0,+\infty) && (15)\\
&\frac{\partial y_2(x,t)}{\partial \nu} = -\mu_1 y_2'(x,t) - \mu_2 z(x,1,t) && \text{on } \Gamma_2\times(0,+\infty), && (16)\\
&y_1(x,t) = y_2(x,t) && \text{on } \Gamma_0\times(0,+\infty), && (17)\\
&a_1\frac{\partial y_1(x,t)}{\partial \nu} = a_2\frac{\partial y_2(x,t)}{\partial \nu} && \text{on } \Gamma_0\times(0,+\infty), && (18)\\
&z(x,0,t) = y'(x,t) && \text{on } \Gamma_2\times(0,+\infty) && (19)\\
&z(x,\rho,0) = f_0(x,-\tau\rho) && \text{on } \Gamma_2\times(0,1) && (20)
\end{aligned}$$

Now, we endow the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega)\times L^2(\Omega)\times L^2(\Gamma_2;L^2(0,1))$$

with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}; \begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{z} \end{pmatrix} \right\rangle = \int_{\Omega} (a(x)\nabla u(x)\nabla \overline{u}(x) + v(x)\overline{v}(x))\, dx + \xi \int_{\Gamma_2}\int_0^1 z(x,\rho)\overline{z}(x,\rho) d\rho\, d\sigma(x)$$

and define a linear operator in $\mathcal{H}$ by

$$\begin{aligned} D(A) = \{&(u,v,z)^T \in H^2(\Omega_1,\Omega_2,\Gamma_1) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2; H^1(0,1); \\ &\frac{\partial u}{\partial \nu} = -\mu_1 v - \mu_2 z(.,1), v = z(.,0) \text{ on } \Gamma_2\} \end{aligned} \tag{21}$$

$$A \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} v \\ a(x)\Delta u \\ -\tau^{-1}\frac{\partial z}{\partial \rho} \end{pmatrix} \tag{22}$$

The spaces used for the definition of $\mathcal{H}$ and $D(A)$ are

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}$$
$$H^2(\Omega_1,\Omega_2,\Gamma_1) = \{u_i \in H^2(\Omega_i) : u = 0 \text{ on } \Gamma_1,\ u_1 = u_2 \text{ and } a_1\frac{\partial u_1}{\partial \nu} = a_2\frac{\partial u_2}{\partial \nu} \text{ on } \Gamma_0\}$$

Then we can rewrite $(12)-(20)$ as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \frac{d}{dt}Y(t) = AY(t) \\ Y(0) = Y_0 \end{cases} \tag{23}$$

where

$$Y(t) = (y, y', z)^T \text{ and } Y_0 = (y_0, y_1, f_0(., -.\tau))^T$$

Proposition 1. *The operator A defined by* (21) *and* (22) *generates a strongly continuous semigroup on $\mathcal{H}$. Thus, for every $Y_0 \in \mathcal{H}$, problem* (23) *has a unique solution Y whose regularity depends on the the initial datum Y_0 as follows:*

$$Y(.) \in C([0,+\infty);\mathcal{H}) \text{ if } Y_0 \in \mathcal{H},$$
$$Y(.) \in C([0,+\infty); D(A)) \cap C^1([0,+\infty);\mathcal{H}) \text{ if } Y_0 \in D(A).$$

Proof. Let $Y = \begin{pmatrix} u \\ v \\ z \end{pmatrix} \in D(A)$. Then

$$\begin{aligned} \langle AY, Y\rangle = &\int_{\Omega} a(x)\nabla u(x).\nabla v(x)dx + \int_{\Omega}(a(x)\Delta u(x))v(x)dx - \\ &\frac{\xi}{\tau}\int_{\Gamma_2}\int_0^1 z_\rho(x,\rho)z(x,\rho)d\rho d\Gamma \end{aligned} \tag{24}$$

Applying Green's first theorem, we obtain

$$\int_{\Omega}(a(x)\Delta u(x))v(x)dx = a_1\int_{\Gamma_1} v(x)\frac{\partial u(x)}{\partial \nu}d\Gamma - a_1\int_{\Omega_1}\nabla u(x).\nabla v(x)dx +$$
$$a_2\int_{\Gamma_2} v(x)\frac{\partial u(x)}{\partial \nu}d\Gamma - a_2\int_{\Omega_2}\nabla u(x).\nabla v(x)dx$$
$$= a_2\int_{\Gamma_2} v(x)\{-\mu_1 v(x) - \mu_2 z(x,1)\}d\Gamma - \int_{\Omega} a(x)\nabla u(x).\nabla v(x)dx \tag{25}$$

Integrating by parts in ρ, we get

$$\int_{\Gamma_2}\int_0^1 z_\rho(x,\rho)z(x,\rho)d\rho d\Gamma = \frac{1}{2}\int_{\Gamma_2}\{z^2(x,1) - z^2(x,0)\}d\Gamma \tag{26}$$

Inserting (25) and (26) into (24) results in

$$\langle AY, Y\rangle = -a_2\mu_1\int_{\Gamma_2} v^2(x)d\Gamma - a_2\mu_2\int_{\Gamma_2} v(x)z(x,1)d\Gamma -$$
$$\frac{\xi}{2\tau}\int_{\Gamma_2} z^2(x,1)d\Gamma + \frac{\xi}{2\tau}\int_{\Gamma_2} v^2(x)d\Gamma$$

from which follows using the Cauchy-Schwarz inequality

$$\langle AY, Y\rangle \le -(a_2\mu_1 - \frac{a_2\mu_2}{2} + \frac{\xi}{2\tau})\int_{\Gamma_2} v^2(x)d\Gamma - (\frac{\xi}{2\tau} - \frac{a_2\mu_2}{2})\int_{\Gamma_2} z^2(x,1)d\Gamma \tag{27}$$

(27) implies that

$$\langle AY, Y\rangle \le 0$$

Thus A is dissipative.

Now we show that for a fixed $\lambda > 0$ and $(g,h,k)^T \in \mathcal{H}$, there exists $Y = (u,v,z)^T \in D(A)$ such that

$$(\lambda I - A)Y = (g,h,k)^T$$

or equivalently

$$\lambda u - v = g \tag{28}$$
$$\lambda v - a(x)\Delta u = h \tag{29}$$
$$\lambda z + \frac{1}{\tau}z_\rho = k \tag{30}$$

Suppose that we have found u with the appropriate regularity, then we can determine z. Indeed, from (21) and (30) we have

$$\begin{cases} z_\rho(x,\rho) = -\lambda\tau z(x,\rho) + \tau k(x,\rho) \\ z(x,0) = v(x) \end{cases}$$

The unique solution of the above initial value problem is

$$z(x,\rho) = e^{-\lambda\tau\rho}v(x) + \tau e^{-\lambda\tau\rho}\int_0^\rho e^{\lambda\tau s}k(x,s)ds$$

and in particular

$$z(x,1) = \lambda e^{-\lambda\tau}u(x) + z_0(x), \quad x \in \Gamma_2$$

where

$$z_0(x) = -e^{-\lambda\tau}g(x) + \tau e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}k(x,s)ds$$

By (28) and (29), the function u satisfies

$$\lambda^2 u - a(x)\Delta u = h + \lambda g \tag{31}$$

Problem (31) can be reformulated as

$$\int_\Omega (\lambda^2 u - a(x)\Delta u)w dx = \int_\Omega (h + \lambda g)w dx, \quad w \in H^1_{\Gamma_1}(\Omega) \tag{32}$$

Using Green's first theorem and recalling (21), we express the right-hand side of (32) as follows

$$\int_\Omega (\lambda^2 u - a(x)\Delta u)w dx = \int_\Omega (\lambda^2 uw + a(x)\nabla u.\nabla w)dx + a_2\int_{\Gamma_2}\{\mu_1(\lambda u - g)w$$
$$+\mu_2(\lambda e^{-\lambda\tau}u(x) + z_0(x))w\}d\Gamma$$

Therefore (32), can be rewritten as

$$\int_\Omega (\lambda^2 uw + a(x)\nabla u.\nabla w)dx + a_2\int_{\Gamma_2}(\mu_1 + \mu_2 e^{-\lambda\tau})\lambda uw d\Gamma = \int_\Omega (h + \lambda g)w d\Gamma$$
$$+a_2\mu_1\int_{\Gamma_2} gw d\Gamma - a_2\mu_2\int_{\Gamma_2} z_0 w d\Gamma, \quad \forall w \in H^1_{\Gamma_1}(\Omega). \tag{33}$$

Since the left-hand side of (33) is coercive on $H^1_{\Gamma_1}(\Omega)$, the Lax-Milgram Theorem guarantees the existence and uniqueness of a solution $y \in H^1_{\Gamma_1}(\Omega)$ of (31). If we consider $w \in \mathcal{D}(\Omega)$ in (28), then y is a solution in $\mathcal{D}'(\Omega)$ of

$$\lambda^2 u - a(x)\Delta u = h + \lambda g \tag{34}$$

and thus $\Delta u \in L^2(\Omega)$.
Combining (33) together with (34), we obtain after using Green's first theorem

$$a_2\int_{\Gamma_2}(\mu_1 + \mu_2 e^{-\lambda\tau})\lambda uw d\Gamma + a_2\int_\Omega \frac{\partial u}{\partial\nu}w d\Gamma = a_2\mu_1\int_{\Gamma_2} gw d\Gamma - a_2\mu_2\int_{\Gamma_2} z_0 w d\Gamma$$

which implies that

$$\frac{\partial u(x)}{\partial\nu} = -\mu_1 v(x) - \mu_2 z(x,1)$$

So, we have found $(u,v,z)^T \in D(A)$ which satisfies $(28) - (30)$. Thus, by the Lumer-Phillips Theorem (see for instance [8], Theorem 1.4.3), generates a strongly continuous semigroup of contractions on $\mathcal{H}$.

3 Proof of Theorem 1

We prove Theorem 1 for smooth initial data. The general case follows by a standard density argument.
We proceed in several steps.
Step 1.
Since

$$E(t) = \frac{1}{2} \|(y, y', z)\|^2_{\mathcal{H}}$$

Then, we deduce from the proof of Proposition 1 that $E(t)$ is non-increasing and

$$\frac{d}{dt}E(t) \leq -C \int_{\Gamma_2} \{y^2(x,t) + y'^2(x,t)\} d\Gamma \tag{35}$$

where

$$C = \min\{a_2\mu_1 - \frac{a_2\mu_2}{2} + \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{a_2\mu_2}{2}\}$$

Step 2.
Set

$$E(t) = \mathcal{E}(t) + E_d(t)$$

where

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} \{a(x) |\nabla y(x,t)|^2 + |y'(x,t)|^2\} dx$$

and

$$E_d(t) = \frac{\xi}{2\tau} \int_{\Gamma_2} \int_0^1 |y'(x, t - \tau\rho)|^2 \, d\rho d\Gamma$$

$E_d(t)$ can be rewritten via a change of variable as

$$E_d(t) = \frac{\xi}{2\tau^2} \int_t^{t+\tau} \int_{\Gamma_2} y'^2(x, s - \tau) d\Gamma ds \tag{36}$$

From (36), we obtain

$$E_d(t) \leq C_1 \int_0^T \int_{\Gamma_2} y'^2(x, s - \tau) d\Gamma ds \tag{37}$$

for $0 \leq t \leq T$ and T large enough.
Step 3.
By applying energy methods (multiplier $2h.\nabla y + (divh - \alpha)y$) (see the appendix) to problem $(1) - (7)$, we obtain for all $T > 0$.

$$\int_0^T \mathcal{E}(t)dt \leq C_2(\mathcal{E}(0) + \mathcal{E}(T)) + C_3 \int_0^T \int_{\Gamma_2} \{(\frac{\partial y(x,t)}{\partial \nu})^2 + y'^2(x,t)\} d\Gamma dt + C_4 \int_0^T \int_{\Gamma_2} |\nabla_\sigma y(x,t)|^2 \, d\Gamma dt + C_5 \int_0^T \int_{\Omega} |y(x,t)|^2 \, d\Omega dt \tag{38}$$

where $\nabla_\sigma y$ is the tangential gradient of y.

Step 4.

We eliminate the tangential gradient from (38) by using the following estimate due to Lasiecka and Triggiani (Lemma 7.2 in [3])

$$\int_\epsilon^{T-\epsilon}\int_{\Gamma_2}|\nabla_\sigma y(x,t)|^2\,d\Gamma dt \leq C_6\{\int_0^T\int_{\Gamma_2}\{(\frac{\partial y(x,t)}{\partial\nu})^2+y'^2(x,t)\}d\Gamma dt+ \|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}\}$$

where ϵ and δ are arbitrary positive constants. We obtain

$$\int_0^T\mathcal{E}(t)dt \leq C_2(\mathcal{E}(0)+\mathcal{E}(T))+C_7\int_0^T\int_{\Gamma_2}\{(\frac{\partial y(x,t)}{\partial\nu})^2+y'^2(x,t)\}d\Gamma dt+ C_8\|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))} \quad (39)$$

Step 5.

We differentiate $\mathcal{E}(t)$ with respect to t and apply Green's first theorem. We obtain

$$\frac{d}{dt}\mathcal{E}(t)=a_2\int_{\Gamma_2}y'(x,t)\frac{\partial y(x,t)}{\partial\nu}d\Gamma dt \quad (40)$$

From (40), we get via the Cauchy-Schwarz inequality

$$\mathcal{E}(0)\leq\mathcal{E}(T)+\frac{a_2}{2}\int_0^T\int_{\Gamma_2}\{y'^2(x,t)+(\frac{\partial y(x,t)}{\partial\nu})^2\}d\Gamma dt \quad (41)$$

Insertion of (41) into (39) yields

$$\int_0^T\mathcal{E}(t)dt\leq 2C_2\mathcal{E}(T)+C_9\int_0^T\int_{\Gamma_2}\{(\frac{\partial y(x,t)}{\partial\nu})^2+y'^2(x,t)\}d\Gamma dt+ C_8\|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))} \quad (42)$$

Step 6.

Since $E(t)$ is non-increasing and $E(t)=\mathcal{E}(t)+E_d(t)$, then (42) together with (37) implies that

$$TE(T)\leq 2C_2\mathcal{E}(T)+C_9\int_0^T\int_{\Gamma_2}\{(\frac{\partial y(x,t)}{\partial\nu})^2+y'^2(x,t)\}d\Gamma dt+ C_8\|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}+TC_1\int_0^T\int_{\Gamma_2}y'^2(x,t-\tau)d\Gamma dt \quad (43)$$

Thus invoking again the identity $E(t)=\mathcal{E}(t)+E_d(t)$ and recalling the boundary condition (4), we obtain from (43)

$$E(T)\leq C_{10}\int_0^T\int_{\Gamma_2}\{y'^2(x,t)+y'^2(x,t-\tau)\}d\Gamma dt+C_{11}\|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))} \quad (44)$$

for T large enough.
Step 7.
We drop the lower order term on the right-hand side of (44) by a compactness-uniqueness argument to obtain

$$E(T) \leq C_{12} \int_0^T \int_{\Gamma_2} \{y'^2(x,t) + y'^2(x,t-\tau)\} d\Gamma dt \tag{45}$$

Step 8.
The estimate (45) together with (35) yields

$$E(T) \leq \frac{C_{13}}{1+C_{13}} E(0) \tag{46}$$

The desired conclusion follows now from (46) since the system (1) − (7) is invariant by translation.

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Appendix: Sketch of Proof of (38)

We multiply both sides of (1) by $2h.\nabla y+(divh-\alpha)y$ and integrate over $(0,T)\times\Omega$. We obtain

$$
\begin{aligned}
&2\int_0^T\int_\Omega a(x)J\nabla y.\nabla y d\Omega dt+\alpha\int_0^T\int_\Omega\{y'^2-a(x)\left|\nabla y\right|^2\}d\Omega dt=\\
&-\int_\Omega\{2y'h.\nabla y+(divh-\alpha)y'y\}_0^T d\Omega-\int_0^T\int_\Omega a(x)y\nabla y.\nabla(divh-\alpha)d\Omega dt+\\
&a_1\int_0^T\int_{\Gamma_1}\left|\frac{\partial y_1}{\partial\nu}\right|^2 h.\nu d\Gamma dt-(a_1-a_2)\int_0^T\int_{\Gamma_0}\left|\nabla y_1\right|^2 h.\nu d\Gamma dt-\\
&\frac{(a_1-a_2)^2}{a_2}\int_0^T\int_{\Gamma_0}\left|\frac{\partial y_1}{\partial\nu}\right|^2 h.\nu d\Gamma dt+\int_0^T\int_{\Gamma_2}\left|y_2'\right|^2 h.\nu d\Gamma dt+\\
&2a_2\int_0^T\int_{\Gamma_2}\left|\frac{\partial y_2}{\partial\nu}\right|^2 h.\nabla y_2 d\Gamma dt-a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2\right|^2 h.\nu d\Gamma dt+\\
&a_2\int_0^T\int_{\Gamma_2}\left|\frac{\partial y_2}{\partial\nu}\right|^2(divh-\alpha)d\Gamma dt
\end{aligned}
\tag{47}
$$

after using the boundary conditions (3) and (5). Identity (47) is used together with $(H.1),(H.2),(H.3)$ and (8) to obtain estimate (38).