Convergence rates for the iteratively regularized Landweber iteration in Banach space

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Abstract. In this paper we provide a convergence rates result for a modified version of Landweber iteration with a priori regularization parameter choice in a Banach space setting.

Keywords: regularization, nonlinear inverse problems, Banach space, Landweber iteration

An increasing number of inverse problems is nowadays posed in a Banach space rather than a Hilbert space setting, cf., e.g., [2, 6, 13] and the references therein.

An Example of a model problem, where the use of non-Hilbert Banach spaces is useful, is the identification of the space-dependent coefficient function c in the elliptic boundary value problem

$$-\Delta u + cu = f \quad \text{in } \Omega \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{2}$$

from measurements of u in $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$, where f is assumed to be known. Here e.g., the choices $p = 1$ for recovering sparse solutions, $q = \infty$ for modelling uniformly bounded noise, or $q = 1$ for dealing with impulsive noise are particulary promising, see, e.g., [3] and the numerical experiments in Section 7.3.3 of [13].

Motivated by this fact we consider nonlinear ill-posed operator equations

$$F(x) = y \tag{3}$$

where F maps between Banach spaces X and Y.

In the example above, the forward operator F maps the coefficient function c to the solution of the boundary value problem (1), (2), and is well-defined as an operator

$$F : \mathcal{D}(F) \subseteq L^p(\Omega) \to L^q(\Omega)\,,$$

* Support by the German Science Foundation DFG under grant KA 1778/5-1 and within the Cluster of Excellence in Simulation Technology (EXC 310/1) at the University of Stuttgart is gratefully acknowledged

where $\mathcal{D}(F) = \{c \in X \mid \exists\, \hat{c} \in L^\infty(\Omega), \hat{c} \geq 0 \text{ a.e. } : \|c - \hat{c}\|_X \leq r\}$, r sufficiently small, for any

$$\begin{array}{ll} p, q \in [1, \infty], \quad f \in L^1(\Omega) & \text{if } d \in \{1, 2\} \\ p \in [1, \infty],\ q \in (\frac{d}{2}, \infty],\ f \in L^s(\Omega),\ s > \frac{d}{2} & \text{if } d \geq 3\,, \end{array}$$

see Section 1.3 in [13].

Since the given data y^δ are typically contaminated by noise, regularization has to be applied. We are going to assume that the noise level δ in

$$\|y - y^\delta\| \leq \delta \tag{4}$$

is known and provide convergence results in the sense of regularization methods, i.e., as δ tends to zero. In the following, x_0 is some initial guess and we will assume that a solution $x^\dagger$ to (3) exists.

Variational methods in Banach space have been extensively studied in the literature, see, e.g., [2, 10, 6] and the references therein.

Since these generalizations of Tikhonov regularization require computation of a global minimizer, iterative methods are an attractive alternative especially for large scale problems. After convergence results on iterative methods for nonlinear ill-posed operator equations in Banach spaces had already been obtained in the 1990's (cf. the references in [1]) in the special case $X = Y$, the general case $X \neq Y$ has only been treaten quite recently, see e.g. [5], [7], and [9] for an analysis of gradient and Newton type iterations. While convergence rates have already been established for the iteratively regularized Gauss-Newton iteration in [7], the question of convergence rates is still open for gradient type, i.e. Landweber methods. It is the aim of this paper to provide such a result.

In order to formulate and later on analyze the method, we have to introduce some basic notations and concepts.

Consider, for some $q \in (1, \infty)$, the duality mapping $J_q^X(x) := \partial \left\{ \frac{1}{q} \|x\|^q \right\}$, which maps from X to its dual X^*. To analyze convergence rates we employ the Bregman distance

$$D_{j_q}(\tilde{x}, x) = \frac{1}{q}\|\tilde{x}\|^q - \frac{1}{q}\|x\|^q - \langle j_q^X(x), \tilde{x} - x \rangle_{X^*.X}$$

(where $j_q^X(x)$ denotes a single valued selection of $J_q^X(x)$) or its shifted version

$$D_q^{x_0}(\tilde{x}, x) := D_{j_q}(\tilde{x} - x_0, x - x_0)\,.$$

Throughout this paper we will assume that X is smooth, which means that the duality mapping is single-valued, and moreover, that X is q-convex, i.e.,

$$D_{j_q}(x, y) \geq c_q \|x - y\|^q \tag{5}$$

for some constant $c_q > 0$. As a consequence, X is reflexive and we also have

$$D_{j_{q^*}}(x^*, y^*) \leq C_{q^*} \|x^* - y^*\|^{q^*}\,, \tag{6}$$

for some C_{q^*}. Here q^* denotes the dual index $q^* = \frac{q}{q-1}$. Moreover, the duality mapping is bijective and $J_q^{-1} = J_{q^*}^{X^*}$, the latter denoting the (by q-convexity also single-valued) duality mapping on X^*. We will also make use of the identities

$$D_{j_q}(x,y) = D_{j_q}(x,z) + D_{j_q}(z,y) + \langle J_q^X(z) - J_q^X(y), x - z\rangle_{X^*,X} \tag{7}$$

and

$$D_{j_q}(y,x) = D_{j_{q^*}}(J_q^X(x), J_q^X(y)) \,. \tag{8}$$

For more details on the geometry of Banach spaces we refer, e.g., to [12] and the references therein.

We here consider the iteratively regularized Landweber iteration

$$\begin{aligned} J_q^X(x_{n+1}^\delta - x_0) &= (1-\alpha_n) J_q^X(x_n^\delta - x_0) - \mu_n A_n^* j_p^Y(F(x_n^\delta) - y^\delta)\,, \\ x_{n+1}^\delta &= x_0 + J_{q^*}^{X^*}(J_q^X(x_{n+1}^\delta - x_0))\,, \qquad n = 0,1,\ldots \end{aligned} \tag{9}$$

where we abbreviate

$$A_n = F'(x_n^\delta)\,,$$

which, for an appropriate choice of the sequence $\{\alpha_n\}_{n\in\mathbb{N}} \in (0,1]$, has been shown to be convergent with rates under a source condition

$$x^\dagger - x_0 \in \mathcal{R}(F'(x^\dagger)^* F'(x^\dagger))^{\nu/2}), \tag{10}$$

with $\nu = 1$ in a Hilbert space setting in [11]. Since the linearized forward operator $F'(x)$ typically has some smoothing property (reflecting the ill-posedness of the inverse problems) condition (10) can often be interpreted as a regularity assumption on the initial error $x^\dagger - x_0$, which is stronger for larger ν.

In the Hilbert space case the proof of convergence rates for the plain Landweber iteration (i.e., (9) with $\alpha_n = 0$) under source conditions (10) relies on the fact that the iteration errors $x_n^\delta - x^\dagger$ remain in the range of $(F'(x^\dagger)^* F'(x^\dagger))^{\nu/2}$ and their preimages under $(F'(x^\dagger)^* F'(x^\dagger))^{\nu/2}$ form a bounded sequence (cf., Proposition 2.11 in [8]). Since carrying over this approach to the Banach space setting would require more restrictive assumptions on the structure of the spaces even in the special case $\nu = 1$, we here consider the modified version with an appropriate choice of $\{\alpha_n\}_{n\in\mathbb{N}} \in (0,1]$.

In place of the Hilbert space source condition (10), we consider variational inequalities

$$\begin{aligned} &\exists\, \beta > 0\ \forall x \in \mathcal{B}_\rho^D(x^\dagger) : \\ &|\langle J_q^X(x^\dagger - x_0), x - x^\dagger\rangle_{X^*\times X}| \le \beta D_q^{x_0}(x^\dagger, x)^{\frac{1-\nu}{2}} \|F'(x^\dagger)(x - x^\dagger)\|^\nu\,, \end{aligned} \tag{11}$$

cf., e.g., [4], where

$$\mathcal{B}_\rho^D(x^\dagger) = \{x \in X \mid D_q^{x_0}(x^\dagger, x) \le \rho^2\}$$

with $\rho > 0$ such that $x_0 \in \mathcal{B}_\rho^D(x^\dagger)$. Using the interpolation and the Cauchy-Schwarz inequality, it is readily checked that in the Hilbert space case (10) implies (11). For more details on such variational inequalities we refer to Section 3.2.3 in [13] and the references therein.

The assumptions on the forward operator besides a condition on the domain

$$\mathcal{B}_\rho^D(x^\dagger) \subseteq \mathcal{D}(F) \tag{12}$$

include a structural condition on its degree of nonlinearity (cf. [4]

$$\begin{aligned}&\left\|(F'(x^\dagger+v)-F'(x^\dagger))v\right\| \leq K \left\|F'(x^\dagger)v\right\|^{c_1} D_q^{x_0}(x^\dagger, v+x^\dagger)^{c_2}\,,\\ &v\in X,\ x^\dagger+v\in\mathcal{B}_\rho^D(x^\dagger)\,,\end{aligned} \tag{13}$$

whose strength depends on the smoothness index in (11). Namely, we assume that

$$c_1 = 1 \text{ or } c_1+c_2p>1 \text{ or } (c_1+c_2p\geq 1 \text{ and } K \text{ is sufficiently small}) \tag{14}$$

$$c_1 + c_2\tfrac{2\nu}{\nu+1} \geq 1\,, \tag{15}$$

so that in case $\nu=1$, a Lipschitz condition on F', corresponding to $(c_1,c_2)=(0,1)$ is sufficient.

Here F' denotes the Gateaux derivative of F, hence a Taylor remainder estimate

$$\begin{aligned}&\left\|F(x_n^\delta)-F(x^\dagger)-F'(x^\dagger)(x_n^\delta-x^\dagger)\right\| \qquad (16)\\ &=\left\|g(1)-g(0)-F'(x^\dagger)(x_n^\delta-x^\dagger)\right\|\\ &=\left\|\int_0^1 g'(t)\,dt - F'(x^\dagger)(x_n^\delta-x^\dagger)\right\|\\ &=\left\|\int_0^1 F'(x^\dagger+t(x_n^\delta-x^\dagger))(x_n^\delta-x^\dagger)\,dt - F'(x^\dagger)(x_n^\delta-x^\dagger)\right\|\\ &\leq K\left\|F'(x^\dagger)(x_n^\delta-x^\dagger)\right\|^{c_1} D_q^{x_0}(x^\dagger,x_n^\delta)^{c_2} \qquad (17)\end{aligned}$$

where $g: t\mapsto F(x^\dagger+t(x_n^\delta-x^\dagger))$, follows from (13).

We will assume that in each step the step size $\mu_n>0$ in (9) is chosen such that

$$\mu_n\frac{1-3C(c_1)K}{3(1-C(c_1)K)}\|F(x_n^\delta)-y^\delta\|^p - 2^{q^*+q-2}C_{q^*}\mu_n^{q^*}\|A_n^* j_p^Y(F(x_n^\delta)-y^\delta)\|^{q^*} \geq 0 \tag{18}$$

where $C(c_1)=c_1^{c_1}(1-c_1)^{1-c_1}$, and c_1, K are as in (13), which is possible, e.g., by a choice $0<\mu_n\leq C_\mu\frac{\|F(x_n^\delta)-y^\delta\|^{\frac{q-p}{q-1}}}{\|A_n\|^{q^*}} =: \overline{\mu}_n$ with $C_\mu := \frac{2^{2-q^*-q}}{3}\frac{1-3C(c_1)K}{(1-C(c_1)K)C_{q^*}}$ If

$$p\geq q \tag{19}$$

and F, F' are bounded on $\mathcal{B}_\rho^D(x^\dagger)$, it is possible to bound $\overline{\mu}_n$ away from zero

$$\overline{\mu}_n \geq C_\mu\left(\sup_{x\in\mathcal{B}_\rho^D(x^\dagger)}(\|F(x)-y\|+\overline{\delta})^{p-q}\|F'(x)\|^q\right)^{-1/(q-1)} =: \underline{\mu} \tag{20}$$

for $\delta \in [0, \overline{\delta}]$, provided the iterates remain in $\mathcal{B}_\rho^D(x^\dagger)$ (which we will show by induction in the proof of Theorem 1). Hence, there exist $\underline{\mu}, \overline{\mu} > 0$ independent of n and δ such that we can choose

$$0 < \underline{\mu} \leq \mu_n \leq \overline{\mu}\,, \tag{21}$$

(e.g., by simply setting $\mu_n \equiv \underline{\mu}$).

Moreover, we will use an a priori choice of the stopping index n_* according to

$$n_*(\delta) = \min\{n \in \mathbb{N} \ : \ \alpha_n^{\frac{\nu+1}{p(\nu+1)-2\nu}} \leq \tau\delta\}\,, \tag{22}$$

and of $\{\alpha_n\}_{n\in\mathbb{N}}$ such that

$$\left(\frac{\alpha_{n+1}}{\alpha_n}\right)^{\frac{2\nu}{p(\nu+1)-2\nu}} + \tfrac{1}{3}\alpha_n - 1 \geq c\alpha_n \tag{23}$$

for some $c \in (0, \frac{1}{3})$ independent of n, where $\nu \in [0,1]$ is the exponent in the variational inequality (11).

Remark 1. A possible choice of $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfying (23) and smallness of $\alpha_{\max}$ is given by

$$\alpha_n = \frac{\alpha_0}{(n+1)^x}$$

with $x \in (0,1]$ such that $3x\theta < \alpha_0$ sufficiently small, since then with $c := \frac{1}{3} - \frac{x\theta}{\alpha_0} > 0$, using the abbreviation $\theta = \frac{2\nu}{p(\nu+1)-2\nu} \in [0, \frac{1}{p-1}]$ we get by the Mean Value Theorem

$$\begin{aligned}
&\left(\frac{\alpha_{n+1}}{\alpha_n}\right)^{\theta} + (\tfrac{1}{3} - c)\alpha_n - 1 \\
&= \frac{\alpha_n}{\alpha_0}\left\{\alpha_0(\tfrac{1}{3} - c) - \frac{(n+2)^{x\theta} - (n+1)^{x\theta}}{(n+2)^{x\theta}}(n+1)^x\right\} \\
&= \frac{\alpha_n}{\alpha_0}\left\{\alpha_0(\tfrac{1}{3} - c) - \frac{x\theta(n+1+t)^{x\theta-1}}{(n+2)^{x\theta}}(n+1)^x\right\} \\
&\geq \frac{\alpha_n}{\alpha_0}\left\{\alpha_0(\tfrac{1}{3} - c) - x\theta\frac{(n+1)^x}{n+1+t}\right\} \geq 0\,,
\end{aligned}$$

for some $t \in [0,1]$.

Theorem 1. *Assume that X is smooth and q-convex, that x_0 is sufficiently close to $x^\dagger$, i.e., $x_0 \in \mathcal{B}_\rho^D(x^\dagger)$, (which by (5) implies that $\|x^\dagger - x_0\|$ is also small), that a variational inequality (11) with $\nu \in (0,1]$ and β sufficiently small is satisfied, that F satisfies (13) with (14), (15), that F and F' are continuous and uniformly bounded in $\mathcal{B}_\rho^D(x^\dagger)$, that (12) holds and that*

$$q^* \geq \frac{2\nu}{p(\nu+1)-2\nu} + 1\,. \tag{24}$$

Let $n_*(\delta)$ be chosen according to (22) with τ sufficiently large. Moreover assume that (19) holds and the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is chosen such that (21) holds for $0<\underline{\mu}<\overline{\mu}$ according to (20), and let the sequence $\{\alpha_n\}_{n\in\mathbb{N}}\subseteq[0,1]$ be chosen such that (23) holds, and $\alpha_{\max}=\max_{n\in\mathbb{N}}\alpha_n$ is sufficiently small.

Then, the iterates x_{n+1}^δ remain in $\mathcal{B}_\rho^D(x^\dagger)$ for all $n\le n_*(\delta)-1$ with n_* according to (22). Moreover, we obtain optimal convergence rates

$$D_q^{x_0}(x^\dagger,x_{n_*})=O(\delta^{\frac{2\nu}{\nu+1}})\,,\quad \text{as } \delta\to 0 \tag{25}$$

as well as in the noise free case $\delta=0$

$$D_q^{x_0}(x^\dagger,x_n)=O\left(\alpha_n^{\frac{2\nu}{p(\nu+1)-2\nu}}\right) \tag{26}$$

for all $n\in\mathbb{N}$.

Remark 2. Note that the rate exponent in (26) $\frac{2\nu}{p(\nu+1)-2\nu}=\frac{2\nu}{\nu+1}(p-\frac{2\nu}{\nu+1})^{-1}$, always lies in the interval $[0,\frac{1}{p-1}]$, since $\frac{2\nu}{\nu+1}\in[0,1]$.

Moreover, note that Theorem 1 provides a results on rates only, but no convergence result without variational inequality. This corresponds to the situation from [11] in a Hilbert space setting.

Proof. First of all, for $x_n^\delta\in\mathcal{B}_\rho^D(x^\dagger)$, (13) allows us to estimate as follows (see also (16)) in case $c_1\in[0,1)$:

$$\begin{aligned}&\left\|F(x_n^\delta)-F(x^\dagger)-A(x_n^\delta-x^\dagger)\right\|\\ &\le K\left\|A(x_n^\delta-x^\dagger)\right\|^{c_1}D_q^{x_0}(x^\dagger,x_n^\delta)^{c_2}\\ &\le C(c_1)K\left(\left\|A(x_n^\delta-x^\dagger)\right\|+D_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{c_2}{1-c_1}}\right)\,,\end{aligned} \tag{27}$$

where we have used the abbreviation $A=F'(x^\dagger)$ and the elementary estimate

$$a^{1-\lambda}b^\lambda\le C(\lambda)(a+b)\text{ with }C(\lambda)=\lambda^\lambda(1-\lambda)^{1-\lambda}\text{ for }a,b\ge 0\,,\ \lambda\in(0,1)\,, \tag{28}$$

and therewith, by the second triangle inequality,

$$\left\|A(x_n^\delta-x^\dagger)\right\|\le\frac{1}{1-C(c_1)K}\left(\left\|F(x_n^\delta)-F(x^\dagger)\right\|+C(c_1)KD_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{c_2}{1-c_1}}\right) \tag{29}$$

as well as analogously

$$\begin{aligned}&\left\|F(x_n^\delta)-F(x^\dagger)-A_n(x_n^\delta-x^\dagger)\right\|\\ &\le 2C(c_1)K\left(\left\|A(x_n^\delta-x^\dagger)\right\|+D_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{c_2}{1-c_1}}\right)\\ &\le\frac{2C(c_1)K}{1-C(c_1)K}\left(\left\|F(x_n^\delta)-F(x^\dagger)\right\|+D_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{c_2}{1-c_1}}\right)\,.\end{aligned} \tag{30}$$

For any $n \leq n_*$ according to (22), by (7) we have

$$\begin{aligned}
&D_q^{x_0}(x^\dagger, x_{n+1}^\delta) - D_q^{x_0}(x^\dagger, x_n^\delta)\\
&= D_q^{x_0}(x_n^\delta, x_{n+1}^\delta) + \langle J_q^X(x_n^\delta - x_0) - J_q^X(x_{n+1}^\delta - x_0), x^\dagger - x_n^\delta\rangle_{X^*\times X}\\
&= D_q^{x_0}(x_n^\delta, x_{n+1}^\delta) - \mu_n\langle j_p^Y(F(x_n^\delta) - y^\delta), A_n(x_n^\delta - x^\dagger)\rangle_{Y^*\times Y}\\
&\quad +\alpha_n\langle J_q^X(x^\dagger - x_0), x^\dagger - x_n^\delta\rangle_{X^*\times X}\\
&\quad -\alpha_n\langle J_q^X(x^\dagger - x_0) - J_q^X(x_n^\delta - x_0), x^\dagger - x_n^\delta\rangle_{X^*\times X} \qquad (31)
\end{aligned}$$

where the terms on the right hand side can be estimated as follows.

By (6) and (8) we have

$$\begin{aligned}
&D_q^{x_0}(x_n^\delta, x_{n+1}^\delta) \qquad (32)\\
&\leq C_{q^*}\|J_q^X(x_{n+1}^\delta - x_0) - J_q^X(x_n^\delta - x_0)\|^{q^*}\\
&= C_{q^*}\|\alpha_n J_q^X(x_n^\delta - x_0) + \mu_n A_n^* j_p^Y(F(x_n^\delta) - y^\delta)\|^{q^*}\\
&\leq 2^{q^*-1}C_{q^*}\left(\alpha_n^{q^*}\|x_n^\delta - x_0\|^q + \mu_n^{q^*}\|A_n^* j_p^Y(F(x_n^\delta) - y^\delta)\|^{q^*}\right)\\
&\leq 2^{q^*-1}C_{q^*}\left(\alpha_n^{q^*}(2^{q-1}(\|x^\dagger - x_0\|^q + \frac{1}{c_q}D_q^{x_0}(x^\dagger, x_n^\delta)) + \mu_n^{q^*}\|A_n^* j_p^Y(F(x_n^\delta) - y^\delta)\|^{q^*}\right) \qquad (33)
\end{aligned}$$

where we have used the triangle inequality in X^* and X, the inequality

$$(a+b)^\lambda \leq 2^{\lambda-1}(a^\lambda + b^\lambda) \quad \text{for } a, b \geq 0\,,\ \lambda \geq 1\,, \qquad (34)$$

and (5).

For the second term on the right hand side of (31) we get, using (30), (28), (34),

$$\begin{aligned}
&\langle j_p^Y(F(x_n^\delta) - y^\delta), A_n(x_n^\delta - x^\dagger)\rangle_{Y^*\times Y}\\
&= \langle j_p^Y(F(x_n^\delta) - y^\delta), F(x_n^\delta) - y^\delta\rangle_{Y^*\times Y}\\
&\quad -\langle j_p^Y(F(x_n^\delta) - y^\delta), F(x_n^\delta) - y^\delta - A_n(x_n^\delta - x^\dagger)\rangle_{Y^*\times Y}\\
&\geq \frac{1-3C(c_1)K}{1-C(c_1)K}\|F(x_n^\delta) - y^\delta\|^p\\
&\quad -\|F(x_n^\delta) - y^\delta\|^{p-1}\left(\frac{2C(c_1)K}{1-C(c_1)K}D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2}{1-c_1}} + \frac{1+C(c_1)K}{1-C(c_1)K}\delta\right)\\
&= \frac{1-3C(c_1)K}{1-C(c_1)K}\|F(x_n^\delta) - y^\delta\|^p\\
&\quad -\left(\frac{1-3C(c_1)K}{3C(\frac{p-1}{p})(1-C(c_1)K)}\|F(x_n^\delta) - y^\delta\|^p\right)^{\frac{p-1}{p}}\left(\frac{(3C(\frac{p-1}{p}))^{p-1}}{(1-C(c_1)K)}\right)^{\frac{1}{p}}\\
&\qquad \left(2C(c_1)KD_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2}{1-c_1}} + (1+C(c_1)K)\delta\right)
\end{aligned}$$

$$\geq \frac{1-3C(c_1)K}{1-C(c_1)K}\|F(x_n^\delta)-y^\delta\|^p - C(\tfrac{p-1}{p})\left\{\frac{1-3C(c_1)K}{3C(\frac{p-1}{p})(1-C(c_1)K)}\|F(x_n^\delta)-y^\delta\|^p \right.$$
$$\left. +\frac{(3C(\frac{p-1}{p})^{p-1}}{(1-C(c_1)K)}2^{p-1}\left((2C(c_1)K)^p D_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{c_2 p}{1-c_1}} + (1+C(c_1)K)^p\delta^p\right)\right\}. \tag{35}$$

Using the variational inequality (11), (29), and

$$(a+b)^\lambda \leq (a^\lambda + b^\lambda) \quad \text{for } a,b\geq 0\,,\ \lambda\in[0,1]\,, \tag{36}$$

we get

$$|\alpha_n\langle J_q^X(x^\dagger-x_0), x^\dagger - x_n^\delta\rangle_{X^*\times X}|$$
$$\leq \beta\alpha_n D_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{1-\nu}{2}}\|F'(x^\dagger)(x_n^\delta - x^\dagger)\|^\nu$$
$$\leq \beta\alpha_n D_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{1-\nu}{2}}\frac{1}{(1-C(c_1)K)^\nu}\left(\|F(x_n^\delta)-y^\delta\| + \delta + C(c_1)KD_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{c_2}{1-c_1}}\right)^\nu$$

$$\leq \beta\alpha_n D_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{1-\nu}{2}}\epsilon^{-\nu}\left(\epsilon\frac{1}{(1-C(c_1)K)^\nu}(\|F(x_n^\delta)-y^\delta\|+\delta)\right)^\nu$$
$$+\beta\alpha_n\left(\frac{C(c_1)K}{(1-C(c_1)K)}\right)^\nu D_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{1-\nu}{2}+\frac{\nu c_2}{1-c_1}}$$
$$\leq C(\tfrac{\nu}{p})\left\{\left(\beta\alpha_n D_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{1-\nu}{2}}\epsilon^{-\nu}\right)^{\frac{p}{p-\nu}} + \left(\epsilon\frac{1}{(1-C(c_1)K)^\nu}(\|F(x_n^\delta)-y^\delta\|+\delta)\right)^p\right\}$$
$$+\beta\alpha_n\left(\frac{C(c_1)K}{(1-C(c_1)K)}\right)^\nu D_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{1-\nu}{2}+\frac{\nu c_2}{1-c_1}}$$

$$= C(\tfrac{\nu}{p})\left\{(\beta\epsilon^{-\nu})^{\frac{p}{p-\nu}}(3C(\tfrac{\nu}{p})C(\tfrac{p(1-\nu)}{2(p-\nu)}))^{\frac{p(1-\nu)}{2(p-\nu)}}\alpha_n^{\frac{p(1+\nu)}{2(p-\nu)}}\left(\frac{\alpha_n D_q^{x_0}(x^\dagger,x_n^\delta)}{3C(\frac{\nu}{p})C(\frac{p(1-\nu)}{2(p-\nu)})}\right)^{\frac{p(1-\nu)}{2(p-\nu)}}\right.$$
$$\left. + \left(\epsilon\frac{1}{(1-C(c_1)K)^\nu}(\|F(x_n^\delta)-y^\delta\|+\delta)\right)^p\right\}$$
$$+\beta\alpha_n\left(\frac{C(c_1)K}{(1-C(c_1)K)}\right)^\nu D_q^{x_0}(x^\dagger,x_n^\delta)^{\frac{1-\nu-c_1+\nu c_1+2\nu c_2}{2(1-c_1)}}$$

$$\leq C(\tfrac{\nu}{p})\left\{C(\tfrac{p(1-\nu)}{2(p-\nu)})\left[\left(\beta\epsilon^{-\nu}(3C(\tfrac{\nu}{p})C(\tfrac{p(1-\nu)}{2(p-\nu)}))^{\frac{1-\nu}{2}}\right)^{\frac{2p}{p(\nu+1)-2\nu}}\alpha_n^{\frac{p(1+\nu)}{p(\nu+1)-2\nu}}\right.\right.$$
$$\left.\left. + \left(\frac{\alpha_n D_q^{x_0}(x^\dagger,x_n^\delta)}{3C(\frac{\nu}{p})C(\frac{p(1-\nu)}{2(p-\nu)})}\right)\right]\right.$$

$$
+\left(\epsilon\frac{1}{(1-C(c_1)K)^\nu}(\|F(x_n^\delta)-y^\delta\|+\delta)\right)^p\Bigg\}
$$
$$
+\frac{1}{3}\alpha_n D_q^{x_0}(x^\dagger,x_n^\delta) \tag{37}
$$

where we have used (28) two times and $\epsilon>0$ will be chosen as a sufficiently small number below. Moreover, by (15), the exponent $\frac{1-\nu-c_1+\nu c_1+2\nu c_2}{2(1-c_1)}=1+\frac{1+\nu}{2(1-c_1)}(c_1+\frac{2\nu}{\nu+1}c_2-1)$ is larger or equal to one and β is sufficiently small so that $\beta\left(\frac{C(c_1)K}{(1-C(c_1)K)}\right)^\nu\rho^{\frac{1-\nu-c_1+\nu c_1+2\nu c_2}{2(1-c_1)}-1}<\frac{1}{3}$.

Finally, we have that

$$
\begin{aligned}
\langle J_q^X(x^\dagger-x_0)-J_q^X(x_n^\delta-x_0),x^\dagger-x_n^\delta\rangle_{X^*\times X} &= D_q^{x_0}(x^\dagger,x_n^\delta)+D_q^{x_0}(x_n^\delta,x^\dagger)\\
&\geq D_q^{x_0}(x^\dagger,x_n^\delta)
\end{aligned} \tag{38}
$$

Inserting estimates (32)-(38) with $\epsilon=2^{p-1}\mu_n^{1/p}\left(\frac{1-3C(c_1)K}{3(1-C(c_1)K)}\right)^{1/p}\frac{(1-C(c_1)K)^\nu}{C(\frac{\nu}{p})}$ into (31) and using boundedness away from zero of μ_n and the abbreviations

$$
\begin{aligned}
\mathrm{d}_n &= D_q^{x_0}(x^\dagger,x_n^\delta)^{1/2}\\
C_0 &= 6^{p-1}C(\tfrac{p-1}{p})^p\frac{(2C(c_1)K)^p}{(1-C(c_1)K)}\\
C_1 &= 2^{q^*+q-2}\frac{C_{q^*}}{c_q}\\
C_2 &= C(\tfrac{\nu}{p})C(\tfrac{p(1-\nu)}{2(p-\nu)})\left(\beta\epsilon^{-\nu}(3C(\tfrac{\nu}{p})C(\tfrac{p(1-\nu)}{2(p-\nu)})^{\frac{1-\nu}{2}}\right)^{\frac{2p}{p(\nu+1)-2\nu}}\\
C_3 &= 2^{q^*+q-2}C_{q^*}\|x^\dagger-x_0\|^q\\
C_4 &= 2^{p-1}C(\frac{\nu}{p})\overline{\epsilon}\frac{1}{(1-C(c_1)K)^\nu}+6^{p-1}C(\tfrac{p-1}{p})^p\frac{(1+C(c_1)K)^p}{1-C(c_1)K}
\end{aligned}
$$

$$
\begin{aligned}
\underline{\epsilon} &= 2^{p-1}\underline{\mu}^{1/p}\left(\frac{1-3C(c_1)K}{3(1-C(c_1)K)}\right)^{1/p}\frac{(1-C(c_1)K)^\nu}{C(\frac{\nu}{p})}\\
\overline{\epsilon} &= 2^{p-1}\overline{\mu}^{1/p}\left(\frac{1-3C(c_1)K}{3(1-C(c_1)K)}\right)^{1/p}\frac{(1-C(c_1)K)^\nu}{C(\frac{\nu}{p})}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\mathrm{d}_{n+1}^2 &\leq C_0\mathrm{d}_n^{\frac{2c_2p}{1-c_1}}+(1-\tfrac{1}{3}\alpha_n+C_1\alpha_n^{q^*})\mathrm{d}_n^2+C_2\alpha_n^{\frac{p(1+\nu)}{p(\nu+1)-2\nu}}+C_3\alpha_n^{q^*}+C_4\delta^p\\
&-\left(\mu_n\frac{1-3C(c_1)K}{3(1-C(c_1)K)}\|F(x_n^\delta)-y^\delta\|^p-2^{q^*+q-2}C_{q^*}\mu_n^{q^*}\|A_n^*j_p^Y(F(x_n^\delta)-y^\delta)\|^{q^*}\right).
\end{aligned}
$$

Here the last term is nonpositive due to the choice (18) of μ_n, so that we arrive at

$$\mathrm{d}_{n+1}^2 \leq C_0 \mathrm{d}_n^{\frac{2c_2p}{1-c_1}} + (1 - \tfrac{1}{3}\alpha_n + C_1\alpha_n^{q^*})\mathrm{d}_n^2 + \underbrace{(C_2 + C_3 + C_4\tau^{-p})}_{=:C_5}\alpha_n^{\frac{p(1+\nu)}{p(\nu+1)-2\nu}} \tag{39}$$

where we have used (24) and the stopping rule (22). Denoting

$$\gamma_n := \frac{\mathrm{d}_n^2}{\alpha_n^{\frac{2\nu}{p(\nu+1)-2\nu}}}$$

we get the following recursion

$$\gamma_{n+1} \leq C_0 \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^\theta \alpha_n^{\theta\omega}\gamma_n^\omega + \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^\theta (1 - \tfrac{1}{3}\alpha_n + C_1\alpha_n^{q^*})\gamma_n + C_5 \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^\theta \alpha_n \tag{40}$$

with

$$\theta = \frac{2\nu}{p(\nu+1)-2\nu} \qquad \omega = \frac{c_2 p}{1-c_1}\,,$$

where

$$\omega \geq 1$$

by (14) and

$$\theta\omega = \frac{p}{p - \frac{2\nu}{\nu+1}} \frac{c_2\frac{2\nu}{\nu+1}}{1-c_1} \geq 1$$

due to assumption (15). Hence as sufficient conditions for uniform boundedness of $\{\gamma_n\}_{n\leq n_*}$ by $\overline{\gamma}$ and for $x_{n+1}^\delta \in \mathcal{B}_\rho^D(x^\dagger)$ we get

$$\overline{\gamma} \leq \rho^2 \tag{41}$$

$$C_0\alpha_n^{\theta\omega-1}\overline{\gamma}^\omega - \left\{\left(\frac{\alpha_{n+1}}{\alpha_n}\right)^\theta + \tfrac{1}{3}\alpha_n - 1 - C_1\alpha_n^{q^*}\right\}\alpha_n^{-1}\overline{\gamma} + C_5 \leq 0\,, \tag{42}$$

where by $q^* > 1$, (15) the factors $C_0\alpha_n^{\theta\omega-1}$, $C_1\alpha_n^{q^*-1}$ and C_5 can be made small for small $\alpha_{\max}$, β, $\|x^\dagger - x_0\|$ and large τ. We use this fact to achieve

$$C_0\alpha_n^{\theta\omega-1}\rho^{\omega-1} + C_1\alpha_n^{q^*-1} \leq \tilde{c} < c$$

with $\tilde{c}$ independent of n, which together with (23) yields sufficiency of

$$\frac{C_5}{c - \tilde{c}} \leq \overline{\gamma} \leq \rho^2$$

for (41), (42), which for any (even small) prescribed ρ is indeed enabled by possibly decreasing β, $\|x^\dagger - x_0\|$, τ^{-1}, and therewith C_5.

In case $c_1 = 1$, estimates (29), (30) simplify to

$$\left\|A(x_n^\delta - x^\dagger)\right\| \le \frac{1}{1-\rho^{2c_2}K}\left\|F(x_n^\delta) - F(x^\dagger)\right\| \tag{43}$$

and

$$\left\|F(x_n^\delta) - F(x^\dagger) - A_n(x_n^\delta - x^\dagger)\right\| \le \frac{2\rho^{2c_2}K}{1-\rho^{2c_2}K}\left\|F(x_n^\delta) - F(x^\dagger)\right\| . \tag{44}$$

Therewith, the terms containing $D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2}{1-c_1}}$ are removed and $C(c_1)$ is replaced by ρ^{2c_2} in (32)-(38), so that we end up with a recursion of the form (40) (with C_0 replace by zero) as before. Hence the remainder of the proof of uniform boundedness of γ_n can be done in the same way as in case $c_1 < 1$.

In case $\delta = 0$, i.e., $n_* = \infty$, uniform boundedness of $\{\gamma_n\}_{n\in\mathbb{N}}$ implies (26). For $\delta > 0$ we get (25) by using (22) in

$$D_q^{x_0}(x^\dagger, x_{n_*}) = \gamma_{n_*}\alpha_{n_*}^{\frac{2\nu}{p(\nu+1)-2\nu}} \le \overline{\gamma}\alpha_{n_*}^{\frac{2\nu}{p(\nu+1)-2\nu}} \le \overline{\gamma}(\tau\delta)^{\frac{2\nu}{\nu+1}}$$

Remark 3. In view of estimate (39), an optimal choice of α_n would be one that minimizes the right hand side. At least in the special case that the same power of α_n appears in the last two terms, i.e., $\frac{p(1+\nu)}{p(\nu+1)-2\nu} = q^*$, elementary calculus yields

$$(\alpha_n^{opt})^{\frac{2\nu}{p(\nu+1)-2\nu}} = \frac{D_q^{x_0}(x^\dagger, x_n^\delta)}{3q^*(C_1 D_q^{x_0}(x^\dagger, x_n^\delta) + C_5)},$$

which shows that the obtained relation $D_q^{x_0}(x^\dagger, x_n^\delta) \sim \alpha_n^{\frac{2\nu}{p(\nu+1)-2\nu}}$ is indeed reasonable and probably even optimal.

References

1. A. B. Bakushinsky and M. Yu. Kokurin. *Iterative methods for approximate solution of inverse problems.* Springer, Dordrecht, 2004.
2. M. Burger and S. Osher. Convergence rates of convex variational regularization. *Inverse Problems*, 20(5):1411–1421, 2004.
3. Christian Clason, Bangti Jin: A semi-smooth Newton method for nonlinear parameter identification problems with impulsive noise, *SIAM J. Imaging Sci* 5:505–538, 2012.
4. T. Hein and B. Hofmann. Approximate source conditions for nonlinear ill-posed problems – chances and limitations. *Inverse Problems*, 25:035003 (16pp), 2009.
5. T. Hein and K. Kazimierski. Accelerated Landweber iteration in Banach spaces. *Inverse Problems*, 26:055002 (17pp), 2010.
6. B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators. *Inverse Problems*, 23(3):987–1010, 2007.
7. B. Kaltenbacher and B. Hofmann. Convergence rates for the iteratively regularized Gauss-Newton method in Banach spaces. *Inverse Problems*, 26:035007 (21pp), 2010.

8. B. Kaltenbacher, A. Neubauer, and O. Scherzer. *Iterative Regularization Methods for Nonlinear Ill-posed Problems.* de Gruyter, 2007.
9. B. Kaltenbacher, F. Schöpfer, and Th. Schuster. Convergence of some iterative methods for the regularization of nonlinear ill-posed problems in Banach spaces. *Inverse Problems*, 25, 2009. 065003 doi: 10.1088/0266-5611/25/6/065003.
10. A. Neubauer, T. Hein, B. Hofmann, S. Kindermann, and U. Tautenhahn. Improved and extended results for enhanced convergence rates of Tikhonov regularization in Banach spaces. *Appl. Anal.*, 89(11):1729–1743, 2010.
11. O. Scherzer. A modified Landweber iteration for solving parameter estimation problems. *Appl. Math. Optim.*, 38:45–68, 1998.
12. F. Schöpfer, A, . K. Louis, and T. Schuster. Nonlinear iterative methods for linear ill-posed problems in Banach spaces. *Inverse Problems*, 22(1):311–329, 2006.
13. T. Schuster, B. Kaltenbacher, B. Hofmann, and K. Kazimierski. *Regularization Methods in Banach Spaces.* de Gruyter, 2007.