A Note on Linear Differential Variational Inequalities in Hilbert Space

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Received: date / Accepted: date

Abstract

Recently a new class of differential variational inequalities has been introduced and investigated in finite dimensions as a new modeling paradigm of variational analysis to treat many applied problems in engineering, operations research, and physical sciences. This new subclass of general differential inclusions unifies ordinary differential equations with possibly discontinuous right-hand sides, differential algebraic systems with constraints, dynamic complementarity systems, and evolutionary variational systems. In this short note we lift this class of nonsmooth dynamical systems to the level of a Hilbert space, but focus to linear input/output systems. This covers in particular linear complementarity systems where the underlying convex constraint set in the variational inequality is specialized to an ordering cone.

The purpose of this note is two-fold. Firstly, we provide an existence result based on maximal monotone operator theory. Secondly we are concerned with stability of the solution set of linear differential variational inequalities. Here we present a novel upper set convergence result with respect to perturbations in the data, including perturbations of the associated linear maps and the constraint set.

1 Introduction

Recently Pang and Stewart [18] introduced and investigated a new class of differential variational inequalities in finite dimensions as a new modeling paradigm of variational analysis to treat many applied problems in engineering, operations research, and physical sciences. This new subclass of general

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differential inclusions unifies ordinary differential equations with possibly discontinuous right-hand sides, differential algebraic systems with constraints, dynamic complementarity systems, and evolutionary variational systems.
Here we lift differential variational inequalities to the more general level of a Hilbert space, but focus to the case of a linear input/output regime, where the operators in the differential equation and in the additional constraint equation are linear. This covers in particular linear complementarity systems, where the underlying convex constraint set in the variational inequality is specialized to an ordering cone. Linear complementarity systems are of much use in mechanical end electrical engineering as well as in optimization [13, 20].
In this note we provide an existence result that relies on maximal monotone operator theory. Furthermore we are concerned with stability of the solution set to differential variational inequalities. In this connection let us refer to [19], where at first several sensitivity results are established for initial value problems of ordinary differential equations with nonsmooth right hand sides and then applied to treat differential variational inequalities. This has to be distinguished from asymptotic Lyapunov stability that has been investigated in [1, 8, 9] for solutions of evolution variational inequalities and nonsmooth dynamical systems. Here we present a novel upper set convergence result with respect to perturbations in the data, including perturbations of the associated linear maps and of the constraint set.

2 Setting of Linear Differential Variational Inequalities

Let X, V be two real, separable Hilbert spaces that are endowed with norms $\| \cdot \|_X$, $\| \cdot \|_V$ respectively and with scalar products denoted by $\langle \cdot, \cdot \rangle, (\cdot, \cdot)$ respectively. Further let there be given $T > 0$, a convex closed subset $K \subset V$, some functions f, g on $[0, T]$ with values in X, respectively in V, and some fixed $x_0 \in X$. Then we consider the following problem: Find an X - valued function x and an V - valued function u both defined on $[0, T]$ that satisfy for a.a. (almost all) $t \in [0, T]$

$$\text{(LDVI)}(\mathcal{A}, f, g, K; x_0) \quad \left\{ \begin{array}{l} \begin{pmatrix} \dot{x}(t) \\ q(t) \end{pmatrix} = \mathcal{A} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \\ u(t) \in K, \quad (q(t), v - u(t)) \geq 0, \ \forall v \in K, \end{array} \right. \tag{2.1}$$

complemented by the initial condition $x(0) = x_0$. Here $\dot{x}(t)$ denotes the time derivative of $x(t)$ and $\mathcal{A} : X \times V \to X \times V$ is a given linear continuous operator that is defined by

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with appropriate linear operators A, B, C, D.
For the closed convex subset K of V and for any $w \in V$, the *tangent cone* (also *support cone* or *contingent cone*, see e.g. [3]) to K at w, denoted by $T_K(w)$, is the closure of the convex cone $\bigcup\{\lambda(K-w) \, : \lambda > 0\}$. Then $T_K(w)$ is clearly a closed convex cone with vertex 0 and is the smallest cone S whose translate $w+S$ has vertex w and contains K. Taking polars with respect to the scalar product in V gives $(T_K(w))^0 = (T_K(w))^- =: N_K(w)$, the normal cone to K at w, which is the subdifferential of the convex indicator function on K; for notions of convex analysis see e.g. [14]. Thus the variational inequality in (2.1) writes as the generalized equation $-q(t) \in N_K(u(t))$.
The fixed finite time interval $[0,T]$ gives rise to the Hilbert space $L^2(0,T;V)$ endowed with the scalar product

$$[u_1, u_2] := \int_0^T (u_1(t), u_2(t)) \, dt, \quad u_1, u_2 \in L^2(0,T;V)\,.$$

Also we introduce the closed convex subset

$$\mathcal{K} := L^2(0,T;K) := \{w \in L^2(0,T;V) \mid w(t) \in K, \forall a.a.\ t \in (0,T)\} \tag{2.2}$$

As in [18] we consider weak solutions of a LDVI in the sense of Caratheodory. In particular, the $X-$ valued function x has to be absolutely continuous with derivative $\dot{x}(t)$ defined almost everywhere. Moreover to define the initial condition, the trace $x(0)$ is needed. Therefore (see [7], Theorem 1, p. 473) we are led to the function space

$$W(0,T;X) := \{x \mid x \in L^2(0,T;X),\ \dot{x} \in L^2(0,T;X)\}\,,$$

a Hilbert space endowed with the scalar product

$$[x_1, x_2] + [\dot{x}_1, \dot{x}_2], \quad x_1, x_2 \in W(0,T;X)\,.$$

Note that $W(0,T;X)$ is continuously and densely embedded in the space $C[0,T;X]$ of X-valued continuous functions on $[0,T]$, where the latter space is equipped with the norm of uniform convergence.

3 Solvability of linear differential variational inequalities

In this section we provide an existence result for linear differential variational inequalities based on maximal monotonicity theory [6, 16]. Here we assume that the given function g is constant, so that shortly $g \in V$.
First we rewrite the variational inequality $(LDVI)_3$ as $-q \in N_K(u)$. By $(LDVI)_2$, $-Cx \in g + Du + N_K(u)$ follows. Hence with the affine map $D_g, D_g v = g + Dv$, we can insert $u \in (D_g + N_K)^{-1}(-Cx)$ in $(LDVI)_1$ and obtain

$$\dot{x} \in Ax + B(D_g + N_K)^{-1}(-Cx) + f. \tag{3.1}$$

Now we adopt an argument due to Brogliato and Goeleven [5] from finite dimension to Hilbert space and assume there exists a coercive selfadjoint operator $P \in \mathcal{L}(X, X)$ such that $B = P\, C^*$. Then P admits a square root $Q \in \mathcal{L}(X, X)$, i.e. $P = Q\, Q^*$ with $Q > 0$ (coercive), hence invertible and therefore $Q^*\, C^* = Q^{-1}\, B$. With $x = -Qz$ (3.1) transforms to

$$\dot{z} \in Q^{-1}AQ\, z - Q^{-1}B(D_g + N_K)^{-1}(CQ\, z) - Q^{-1}f. \tag{3.2}$$

Let us assume that $D \geq 0$, i.e. $(D\, v, v) \geq 0$. Then $D_g + N_K$ is maximal monotone by [6, Proposition 2.4, Corollaire 2.7]. Clearly, also the inverse $(D_g + N_K)^{-1}$ is maximal monotone.
Furthermore we use the notion of the relative interior denoted by rint and assume the regularity condition $0 \in \text{rint}\, \Big[\text{im}\, (C\, Q) - \text{dom}\, ((D_g + N_K)^{-1})\Big]$. Then by [17, Cor. 4.4], [21, Theorem 4], also $Q^{-1}B(D_g + N_K)^{-1}CQ$ is maximal monotone in virtue of $(C\, Q)^* = Q^{-1}B$. Since $Q^{-1}AQ$ is a Lipschitz perturbation, [6, Theorem 3.17; Corollaire 3.2], [16, Theorem 2.1, Remark 2.1] applies to conclude the existence of a unique strong solution $z \in W^{1,\infty}(0, T; X)$ to (3.2) with $z(0) = -Q^{-1}x_0$, provided $f \in W^{1,1}(0, T; X)$ and $z_0 := -Q^{-}1\, x_0$ satisfies $CQz_0 \in \text{dom}\, (D_g + N_K)^{-1} = \text{im}\, (D_g + N_K)$.
If moreover $D > 0$ with a coercivity constant $\delta > 0$, then from the variational inequality $(LDVI)_3$ we get uniqueness of u and the estimate

$$\|u(s) - u(t)\|_V \leq \frac{\|C\|}{\delta} \|x(s) - x(t)\|_X \quad s, t \in [0, T]\,,$$

that shows that u is $W^{1,\infty}$ on $(0, T)$, too.
Thus we have proven the following existence result.

Theorem 3.1 *Suppose $D \geq 0$ and there exists $P = P^* > 0$ such that $B = PC^*$. Moreover assume the regularity condition $0 \in$ rint $\Big[$im $(C\ Q) -$*

dom $((D+N_K)^{-1})\big]$, *where* $P = Q\ Q^*$. *Then for any* $f \in W^{1,1}(0,T;X)$, $g \in V$, *and for any* x_0 *such that* $-Cx_0 - g \in$ *im* $(D+N_K)$, *(LDVI) is uniquely solvable with* $x \in W^{1,\infty}(0,T;X)$ *and* $x(0) = x_0$. *- If moreover* $D > 0$, *then* u *is unique, too, and* $u \in W^{1,\infty}(0,T;V)$.

Remark. Let $\mathcal{M}$ be a general maximal monotone map that replaces the above normal cone map N_K. Then by a similar reasoning as above we obtain an existence result for the multivalued Luré dynamical system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + f(t);\ x(0) = x_0 \\ u(t) \in \mathcal{M}[Cx(t) + Du(t)]. \end{cases}$$

Luré dynamical systems with $\mathcal{M} = \partial\varphi$, φ a convex closed and proper function have been recently studied by Brogliato and Goeleven [5] in finite dimensions with applications in nonsmooth electronics.

4 Stability of linear differential variational inequalities

In this section we study stability of linear differential variational inequalities formulated as LDVI and admit perturbations $x_{0,n}$ of x_0 in the initial condition $x(0) = x_0$, $\mathcal{A}_n = (A_n, B_n, C_n, D_n)$ of the linear map $\mathcal{A} = (A, B, C, D)$, f_n, g_n of the functions f, g, and K_n of the convex closed subset $K \subset V$. Suppose that (x^n, u^n) solves (LDVI)$(\mathcal{A}_n, f_n, g_n, K_n; x_{0,n})$ and assume that $(x^n, u^n) \to (x, u)$ with respect to an appropriate convergence for X-valued, respectively V-valued functions on $[0,T]$. Then we seek conditions on $\mathcal{A}_n \to \mathcal{A}, f_n \to f, g_n \to g, K_n \to K, x_{0,n} \to x_0$ that guarantee that (x, u) solves the limit problem (LDVI)$(\mathcal{A}, f, g, K; x_0)$. Such a stability result can be understood as a result of upper set convergence for the solution set of the LDVI.

4.1 Preliminaries; Mosco convergence of sets

As the convergence of choice in variational analysis we employ Mosco set convergence for a sequence $\{K_n\}$ of closed convex subsets which is defined as follows. A sequence $\{K_n\}$ of closed convex subsets of the Hilbert space V is called Mosco convergent to a closed convex subset K of V, written $K_n \xrightarrow{M} K$, if and only if

$$\sigma - \limsup_{n\to\infty} K_n \subset K \subset s - \liminf_{n\to\infty} K_n.$$

Here the prefix σ means sequentially weak convergence in contrast to strong convergence denoted by the prefix s; $\limsup$, respectively $\liminf$ are in the

sense of Kuratowski upper, resp. lower limits of sequences of sets (see [2, 4] for more information on Mosco convergence).
As a preliminary result we need that Mosco convergence of convex closed sets K_n inherits to Mosco convergence of the associated sets $\mathcal{K}_n = L^2(0,T;K_n)$, derived from K_n similar to (2.2).

Lemma 4.1 *Let $K_n \stackrel{M}{\longrightarrow} K$. Then $\mathcal{K}_n \stackrel{M}{\longrightarrow} \mathcal{K}$ in $L^2(0,T;V)$.*

For the proof we refer to [10, 12].
As a further tool in our stability analysis we recall from [11] the following technical result.

Lemma 4.2 *Let H be a separable Hilbert space and let $T > 0$ be fixed. Then for any sequence $\{z_n\}_{n\in\mathbf{N}}$ converging to some z in $L^1(0,T;H)$ there exists a subsequence $\{z_{n_k}\}_{k\in\mathbf{N}}$ such that for some set N of zero measure, $z_{n_k}(t) \stackrel{s}{\longrightarrow} z(t)$ for all $t \in [0,T]\backslash N$.*

4.2 The stability result

We need the following hypotheses on the convergence of the perturbations:
(H1) Convergence $\mathcal{A}_n \to \mathcal{A}$ holds in the operator norm topology. - All operators D_n are monotone, i.e. for any $v \in V$, $(D_n v, v) \geq 0$ holds.
(H2) Convergence of the functions $f_n \to f, g_n \to g$ holds in $L^2(0,T;X)$, respectively in $L^2(0,T;V)$.
Now we can state the following stability result.

Theorem 4.1 *Let (x^n, u^n) solve (LDVI)$(\mathcal{A}_n, f_n, g_n, K_n; x_{0,n})$. Suppose, $\mathcal{A}_n$ and $\mathcal{A}$ satisfy (H1), and that f_n, g_n and f, g satisfy (H2). Let the convex closed sets K_n Mosco-converge to K and let $x_{0,n} \stackrel{s}{\to} x_0$. Assume that $x^n \stackrel{s}{\to} x$ in $W(0,T;X)$ and that $u^n \in L^2(0,T;V)$ converges weakly to u pointwise in V for a.a. $t \in (0,T)$ with $\|u^n(t)\|_V \leq m(t)$, $\forall$ a.a. $t \in (0,T)$ for some $m \in L^2(0,T)$. Then (x,u) is a solution to (LDVI)$(\mathcal{A}, f, g, K; x_0)$.*

Proof.
The proof consists of three parts.
1. Feasibility: $u \in \mathcal{K}, x(0) = x_0$.
First we observe that for any $w \in L^2(0,T;V)$, in virtue of Lebesgue's theorem of dominated convergence,

$$[u^n, w] = \int_0^T (u^n(t), w(t))\; dt \to [u, w]\,.$$

Thus $u^n \xrightarrow{\sigma} u$ and $u \in L^2(0,T;V)$. Moreover directly by Mosco convergence of $\{K_n\}$ or invoking lemma 4.1, $u \in \mathcal{K}$ follows. - Since by continuous embedding $x^n \xrightarrow{s} x$ in $C[0,T;X]$, we conclude $x^n(0) = x_{0,n} \xrightarrow{s} x(0) = x_0$.
2. u solves the variational inequality in (LDVI)$(\mathcal{A}, f, g, K; x_0)$:
Fix an arbitrary $w \in \mathcal{K}$. Then by lemma 4.1, there exist $w^n \in \mathcal{K}_n$ such that $w^n \xrightarrow{s} w$ in $L^2(0,T;V)$. Moreover, by extracting eventually a subsequence, we have by lemma 4.2 that $w^n(t), g_n(t)$ strongly converges to $w(t), g(t)$,respectively, for a.a. $t \in (0,T)$. For any measureable set $A \subset (0,T)$ we can define $w^n_A \in L^2(0,T;V)$ by $w^n_A = w^n$ on A, $w^n_A = u^n$ on $(0,T) \setminus A$. Hence $w^n_A \in \mathcal{K}_n$ and by construction,

$$\int_A (q^n(t), w^n(t) - u^n(t))\; dt \;\geq\; 0\,,$$

where $q^n(t) = C_n\; x^n(t) + D_n\; u^n(t) + g_n(t)$. Hence a contradiction argument shows that we have pointwise for a.a. $t \in (0,T)$, $(q^n(t), w^n(t) - u^n(t)) \geq 0\,$. By (H1), monotonicity entails $(C_n\; x^n(t) + D_n\; w^n(t) + g_n(t), u^n(t) - w^n(t)) \leq 0\,$. By (H1) and (H2), in the limit $(C\; x(t) + D\; w(t) + g(t), u(t) - w(t)) \;\leq\; 0\,$. In virtue of the linear growth of the linear operators we arrive at

$$[G(x,w), u - w] := \int_0^T (C\; x(t) + D\; w(t) + g(t), u(t) - w(t))\; dt \;\leq\; 0,\, \forall w \in \mathcal{K}\,.$$

Hence by a well-known argument in monotone operator theory (see e.g. [22]) we obtain that $u \in \mathcal{K}$ satisfies the variational inequality

$$[G(x,u), w - u] \geq 0,\, \forall w \in \mathcal{K}\,.$$

3. (x,u) solves the limit problem (LDVI)$(\mathcal{A}, f, g, K; x_0)$:
By Lemma 4.2 applied to $\{f_n\}$, $\{x^n\}$, and $\{\dot{x}^n\}$, we can extract a subsequence such that $f_n(t) \to f(t)$, $x^n(t) \to x(t)$, and $\dot{x}^n(t) \to \dot{x}(t)$ strongly in X pointwise for all $t \in (0,T)\backslash N_0$, where N_0 is a null set. Fix $t \in (0,T)\backslash N_0$. Then by assumption, for all $n \in \mathbf{N}$ we have $\dot{x}^n(t) = A_n\; x^n(t) + B_n\; u^n(t) + f_n(t)$. Then in virtue of (H1) and (H2), $\dot{x}(t) = A\; x(t) + B\; u(t) + f(t)$ follows and (x,u) solves (LDVI)$(\mathcal{A}, f, g, K; x_0)$. ∎

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