On the Normal Semilinear Parabolic Equations Corresponding to 3D Navier-Stokes System

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Abstract. The semilinear normal parabolic equations corresponding to 3D Navier-Stokes system have been derived. The explicit formula for solution of normal parabolic equations with periodic boundary conditions has been obtained. It was shown that phase space of corresponding dynamical system consists of the set of stability (where solutions tends to zero as time $t \to \infty$), the set of explosions (where solutions blow up during finite time) and intermediate set. Exact description of these sets has been given.

Keywords: Equations of normal type, Navier-Stokes system, structure of dynamical flow

1 Introduction

As well known (see e.g. [1],[2]), existence of weak solution to 3D Navier-Stokes equations is proved with help of energy estimate, which is true because the image $B(v)$ of nonlinear operator from Navier-Stokes equations consists of vectors tangent to sphere in the L_2-space with the centrum in origin. If these vectors would be tangent to sphere in Soblev H^1-space, one could prove existence of strong solution to 3D Navier-Stokes system by the methods similar to ones used to prove existence of a weak solution. But this is not the matter: in this case $B(v) = B_\tau(v)+B_n(v)$ where $B_\tau(v)$ is the component tangent to sphere in H^1 and $B_n(v)$ is normal component. In this paper we change nonlinear operator $B(v)$ of input system on its normal part $B_n(v)$. Obtained equations, which we call Normal Parabolic Equations do not satisfy to analog in H^1 of energy estimate "'in the most degree"'. We hope that investigation of these equations can help to understand better the problems connected with solvability of 3d Navier-Stokes system in the class of strong solutions.

In this paper we study Normal Parabolic Equations (NPE) corresponding to 3D Navier-Stokes system. In Section 2 we derive NPE. In Section 3 we study some properties of NPE. The key property obtained there is existence of explicit formula for solution to NPE. In section 4 the structure of dynamical flow corresponding to NPE is investigated.

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Note that NPE has been introduced in [3] where normal parabolic equation corresponding to Burgers equation was studied. Here we generalize results from [3] on the case of NPE corresponding to Navier-Stokes system and, besides, continuer to develop the theory of NPE.

2 Semilinear Parabolic Equations of Normal Type

Our aim is to try to understand better how to investigate 3D Navier-Stokes system in phase space of one time differentiable vector fields where energy estimate is not true. To this end we derive some semilinear parabolic equation.

2.1 Navier-Stokes System and Helmholtz Equations

Let consider 3D Navier-Stokes equations with periodic boundary conditions:

$$\partial_t v(t,x) - \Delta v + (v,\nabla)v + \nabla p(t,x) = 0, \quad \operatorname{div} v = 0, \tag{1}$$

$$v(t,\ldots,x_i,\ldots) = v(t,\ldots,x_i+2\pi,\ldots), \qquad i=1,2,3, \tag{2}$$

$$v(t,x)|_{t=0} = v_0(x) \tag{3}$$

where $t \in \mathbb{R}_+, x=(x_1,x_2,x_3) \in \mathbb{R}^3,\ v(t,x)=(v_1,v_2,v_3)$ is the velocity vector field of fluid flow, ∇p is the gradient of pressure, Δ is Laplace operator, $(v,\nabla)v = \sum_{j=1}^3 v_j \partial_{x_j} v$. Periodic boundary conditions (2) mean in fact that Navier-Stokes equations (1) and initial conditions (3) are defined on torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$.

We transform problem (1)-(3) for velocity to the problem for curl of velocity as unknown function:

$$\omega(t,x) = \text{curl } v(t,x) = (\partial_{x_2}v_3 - \partial_{x_3}v_2,\ \partial_{x_3}v_1 - \partial_{x_1}v_3,\ \partial_{x_1}v_2 - \partial_{x_2}v_1) \tag{4}$$

Recall the following well-known formulas of vectorial analysis:

$$(v,\nabla)v = \omega \times v + \nabla \frac{|v|^2}{2}, \tag{5}$$

$$\text{curl } (\omega \times v) = (v,\nabla)\omega - (\omega,\nabla)v, \quad \text{if} \quad \operatorname{div} v = 0, \quad \operatorname{div}\omega = 0 \tag{6}$$

where $\omega \times v = (\omega_2 v_3 - \omega_3 v_2,\ \omega_3 v_1 - \omega_1 v_3,\ \omega_1 v_2 - \omega_2 v_3)$ is vector product and $|v|^2 = v_1^2 + v_2^2 + v_3^2$. Let substitute (5) into the first equality of (1) and apply to both parts of obtained equality operator curl. Then in virtue of (4),(6), and formula $\text{curl}\nabla F = 0$ we obtain the Helmholtz equations

$$\partial_t \omega(t,x) - \Delta\omega + (v,\nabla)\omega - (\omega,\nabla)v = 0 \tag{7}$$

We add these equations with initial conditions

$$\omega(t,x)|_{t=0} = \omega_0(x) \tag{8}$$

where $\omega_0 = \text{curl } v_0$.

2.2 Normal Parabolic Equations (NPE) and Their Derivation

For each $m \in \mathbb{Z}_+ = \{j \in \mathbb{Z} : j \geq 0\}$ we define the space

$$V^m = V^m(\mathbb{T}^3) = \{v(x) \in (H^m(\mathbb{T}^3))^3 : \mathrm{div} v = 0,\ \int_{\mathbb{T}^3} v(x)dx = 0\} \tag{9}$$

where $H^m(\mathbb{T}^3)$ is Sobolev space.

Multiplying Navier-Stokes system (1) on v scalarly in $L_2(\mathbb{T}^3)$ we obtain after integration by parts (on x) and integration on t well-known energy estimate

$$\int_{\mathbb{T}^3} |v(t,x)|^2 dx + 2\int_0^t \int_{\mathbb{T}^3} |\nabla_x v(\tau,x)|^2 dx d\tau \leq \int_{\mathbb{T}^3} |v_0(x)|^2 dx \tag{10}$$

that gives opportunity to prove existence of weak solutions to problem (1)-(3). Unfortunately, this solution is not smooth enough to establish its uniqueness. If in a hope to get existence of smooth solution to (1) we would try to get analog of energy estimate in phase space V^1, multiplying (1) on v scalarly in $V^1(\mathbb{T}^3)$ we will not get analog of bound (10). Let try to understand situation passing from Navier-Stokes to Helmholtz equations.

Using decomposition in Fourier series

$$v(x) = \sum_{k \in \mathbb{Z}^3} \widehat{v}(k) e^{ix\cdot k}, \quad \text{where} \quad \widehat{v}(k) = (2\pi)^{-3} \int_{\mathbb{T}^3} v(x) e^{-ix\cdot k} dx,$$

$x \cdot k = \sum_{j=1}^3 x_j k_j,\ k = (k_1, k_2, k_3)$, and well-known formula curl curl $v = -\Delta v$ if $\mathrm{div}\, v = 0$, we see that on space V^m inverse operator to curl is well-defined and is determined by the formula

$$\mathrm{curl}^{-1}\omega(x) = i \sum_{k \in \mathbb{Z}^3} \frac{k \times \widehat{\omega}(k)}{|k|^2} e^{ix\cdot k} \tag{11}$$

That is why operator

$$\mathrm{curl}\ :\ V^1 \longrightarrow V^0$$

realized isomorphism of the spaces. Therefore sphere in V^1 for problem (1), (3) is equivalent to sphere in V^0 for problem (7), (8).

Let denote nonlinear term of Helmholtz equation by B:

$$B(\omega) = (v, \nabla)\omega - (\omega, \nabla)v \tag{12}$$

(we did not indicate dependence B on v because it can be expressed via ω by (11)).

Multiplying equality (12) on $\omega = (\omega_1, \omega_2, \omega_3)$ scalarly in V^0 and integrating by parts we get

$$(B(\omega), \omega)_{V^0} = -\int_{\mathbb{T}^3} \sum_{j,k=1}^3 \omega_j \partial_j v_k \omega_k dx \tag{13}$$

that, generally saying, is not equal to zero. Just because of this 3D Helmholtz equations do not possess energy estimate. In other words, operator B admits the decomposition

$$B(\omega) = B_n(\omega) + B_\tau(\omega) \tag{14}$$

where vector $B_n(\omega)$ is orthogonal to the sphere $\Sigma_\omega = \{u \in V^0 : \|u\|_{V^0} = \|\omega\|_{V^0}\}$ at the point ω, and vector $B_\tau(\omega)$ is tangential to Σ_ω at ω. Generally saying, both operators B_n, B_τ in (14) are not equal to zero. Note that just the component $B_n \neq 0$ prevents to derivation of energy bound and therefore it is quite possible that the main difficulties obstructing to investigation of Navier-Stokes equations are connected just with this operator. That is why there is reason to omit in Helmholtz equations the component B_τ and to study on the first stage analog of equations (7) in which nonlinear operator $B(\omega)$ is changed on its normal component $B_n(\omega)$. Such equations we call Normal Parabolic Equations (NPE).

Let construct now normal parabolic equations with respect to sphere in V^0, corresponding to problem (7), (8).

Since summand $(v, \nabla)\omega$ from (7) is tangential operator:

$$\int_{\mathbb{T}^3} (v, \nabla)\omega \cdot \omega dx = 0,$$

normal part of nonlinear operator from (7) is defined by nonlinear term $(\omega, \nabla)v$. We look it for in the form $\Phi(\omega)\omega$ where Φ is unknown functional that is found by equation

$$\int_{\mathbb{T}^3} \Phi(\omega)\omega(x) \cdot \omega(x) dx = \int_{\mathbb{T}^3} (\omega(x), \nabla)v(x) \cdot \omega(x) dx \tag{15}$$

Relation (15) implies desired formula for Φ:

$$\Phi(\omega) = \begin{cases} \int_{\mathbb{T}^3} (\omega(x), \nabla)\mathrm{curl}\,^{-1}\omega(x) \cdot \omega(x) dx \,/ \int_{\mathbb{T}^3} |\omega(x)|^2 dx, & \omega \neq 0, \\ 0, & \omega \equiv 0 \end{cases} \tag{16}$$

where $\mathrm{curl}\,^{-1}\omega(x)$ is defined in (11).

So, normal parabolic equations corresponding to system (7) are defined as follows:

$$\partial_t \omega(t, x) - \Delta\omega - \Phi(\omega)\omega = 0, \quad \mathrm{div}\,\omega = 0 \tag{17}$$

where functional Φ is defined in (16). These equations supplied with initial conditions (8) and periodic boundary conditions are the main object of our investigation in this paper.

3 Properties of Normal Parabolic Equations

3.1 Explicit Formula for Solution of NPE

In this subsection we derive explicit formula for NPE solution. This is the key result because it gives the possibility to establish many important properties on NPE. Some of them will be obtained below in next sections. The following assertion is true:

Lemma 1. *Let $S(t,x,y_0)$ be resolving operator of the following Stokes system with periodic boundary conditions:*

$$\partial_t y(t,x) - \Delta y(t,x) = 0, \quad div\, y = 0, \quad y(t,x)|_{t=0} = y_0(x), \tag{18}$$

i.e. $S(t,x,y_0) = y(t,x)$ (we assume, of course, that $div\, y_0 = 0$). The solution of problem (17),(8) has the form

$$\omega(t,x;\omega_0) = \frac{S(t,x;\omega_0)}{1 - \int_0^t \Phi(S(\tau,x;\omega_0))d\tau} \tag{19}$$

The proof of this lemma is reduced to substitution (19) into (17) and direct checking of obtained equality.

3.2 Properties of Functional Φ

Let $s \in \mathbb{R}$. Recall that Sobolev space $H^s(\mathbb{T}^3)$ is the space of periodic real-valued distributions $z(x)$ possessing with the finite norm

$$\|z\|^2_{H^s(\mathbb{T}^3)} \equiv \|z\|^2_s = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} |k|^{2s}|\widehat{z}(k)|^2 < \infty \tag{20}$$

where $\widehat{z}(k)$ are Fourier coefficients of z.

We will use the following generalization of spaces (9) of solenoidal vector fields:

$$V^s \equiv V^s(\mathbb{T}^3) = \{v(x) \in (H^s(\mathbb{T}^3))^3 : \quad \mathrm{div} v(x) = 0, \quad \int_{\mathbb{T}^3} v(x)dx = 0\}, \quad s \in \mathbb{R} \tag{21}$$

Lemma 2. *Let $\Phi(u)$ be functional (16). There exists a constant $c > 0$ such that for each $u \in V^{3/2}$*

$$|\Phi(u)| \le c\|u\|_{3/2} \tag{22}$$

This lemma is proved similarly to analogous bound from [3].

Lemma 3. *Let Φ be functional (16). For each $\beta < 1/2$ there exists a constant $c_1 > 0$ such that for each $y_0 \in V^{-\beta}(\mathbb{T}^3),\ t > 0$*

$$\left| \int_0^t \Phi(S(\tau,\cdot,y_0))d\tau \right| \le c_1\|y_0\|_{-\beta} \tag{23}$$

where $S(t,\cdot,y_0)$ is resolving operator of problem (18).

Proof. Using (22) we get

$$\left| \int_0^t \Phi(S(\tau,\cdot,y_0))d\tau \right| \le c\int_0^t e^{-\tau/2} \left(\sum_{k\ne 0} (|\widehat{y}_0(k)|^2|k|^{-2\beta})|k|^{3+2\beta} e^{-(k^2-1)\tau} \right)^{1/2} d\tau \tag{24}$$

where $\widehat{y}_0(k)$ are Fourier coefficients of y_0. Solution $\hat{\rho} = \rho(t)$ of extremal problem

$$f(t,\rho) = \rho^{3+2\beta} e^{-(\rho^2-1)t} \to \max, \quad \rho \geq 1$$

is defined with expression $\rho(t) = \sqrt{\frac{3+2\beta}{2t}}$, and

$$f(t,\rho(t)) = \begin{cases} \left(\frac{3+2\beta}{2t}\right)^{\frac{3+2\beta}{2}} e^{-(3+2\beta-2t)/2}, & t \leq \frac{3+2\beta}{2}, \\ 1, & t \geq \frac{3+2\beta}{2} \end{cases} \tag{25}$$

Substitution (25) into (24) implies (23).

Remark 1. Lemma 3 implies that the functional from left side of bound (23) is well defined for $y_0 \in V^{-\beta}(\mathbb{T}^3)$ with $\beta < 1/2$. In particular, in virtue of this Lemma and (19) solution of problem (17),(8) is well defined for each initial condition $\omega_0 \in V^0$, and therefore our choice V^0 as phase space for corresponding dynamical system is correct. Note also that simple modification of Lemma 3 proof gives continuity of the functional from left side in (23) with respect to $y_0 \in V^{-\beta},\ \beta < 1/2$.

4 The Structure of NPE Dynamics

The aim of this section is to find out the main feature of dynamical flow corresponding to NPE. We decompose the phase space of the dynamical system on three sets with different behavior of dynamical flow inside each of them.

4.1 Distinctive Sets of Phase Space

Let give definitions of three subsets of phase space for NPE. Recall that we take $V^0(\mathbb{T}^3) \equiv V^0$ as the phase space for problem (17),(8).

Definition 1. *The set $M_- \equiv M_-(\alpha) \subset V^0$ of initial conditions ω_0 such that the solution $\omega(t,x;\omega_0)$ of problem (17),(8) exists and satisfies inequality*

$$\|\omega(t,\cdot;\omega_0)\|_0 \leq \alpha \|\omega_0\|_0 e^{-t} \qquad \forall t > 0 \tag{26}$$

is called the set of stability. Here $\alpha > 1$ is a certain fixed number.

The following simple sufficient condition for belonging to $M_-(\alpha)$ is true in virtue of (19): If $\omega_0 \in V^0$ satisfies the bound

$$\sup_{t \in \mathbb{R}_+} \int_0^t \Phi(S(\tau,\cdot;\omega_0))d\tau \leq \frac{\alpha - 1}{\alpha} \tag{27}$$

then $\omega_0 \in M_-(\alpha)$.

Definition 2. *The set $M_+ \subset V^0$ of initial conditions ω_0 from (17),(8) such that corresponding solution $\omega(t, x; \omega_0)$ exists only on a finite interval $t \in (0, t_0)$ with $t_0 > 0$ depending on ω_0, and blows up at $t = t_0$ is called the set of explosions.*

In virtue of formula (19) for solution $\omega(t, x; \omega_0)$

$$M_+ = \{\omega_0 \in V^0 : \ \exists t_0 > 0 \ \int_0^{t_0} \Phi(S(\tau, \cdot; \omega_0))d\tau = 1\} \tag{28}$$

The minimal magnitude from the set $\{t_0\}$ for which equality in (28) holds is called the time of explosion.

Definition 3. *The collection*

$$M_I(\alpha) = V^0 \setminus \{M_-(\alpha) \cup M_+\} \tag{29}$$

is called intermediate set.

Remark 2. Definitions of stability and intermediate sets include parameter $\alpha > 1$ and from this point of view they are not absolute. Nevertheless they are convenient for using.

We study below the properties of these sets and, in particular, we show that all these sets are nonempty. We begin from the set of stability. This set is the most important for us.

4.2 Subsets Belonging to the Set of Stability

Let $\rho > 0, \beta < 1/2$. Introduce the set

$$\begin{aligned} El_\rho^\beta &= \{v \in V^0(\mathbb{T}^3) : \ \|v\|_{-\beta}^2 = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|\widehat{v}(k)|^2}{|k|^{2\beta}} \le \rho^2\} \\ &= \{v \in V^0(\mathbb{T}^3) : \sum_{k \in \mathbb{Z}^3 \setminus \{0\}}^{\infty} \frac{|\widehat{v}(k)|^2}{\rho^2 |k|^{2\beta}} \le 1\}, \end{aligned} \tag{30}$$

which we can interpret as ellipsoid in $V^0(\mathbb{T}^3)$ with length of axes directed along functions $e^{ik \cdot x}, e^{-ik \cdot x}$ equal to $\rho |k|^\beta$. Since $\rho |k|^\beta \to \infty$ as $|k| \to \infty$, this ellipsoid is unbounded in V^0.

Lemma 4. *Let $c_1 \rho < 1$ and $\rho \le (\alpha - 1)/(\alpha c_1)$ where c_1 is the constant from (23). Then*

$$El_\rho^\beta \subset M_-(\alpha) \tag{31}$$

where the set El_ρ^β is defined in (30).

Proof. Note that solution $S(t,x,\omega_0)$ of problem (18) with $y_0=\omega_0$ satisfies inequality:

$$\|S(t,\cdot,\omega_0)\|_0^2=\sum_{k\neq 0}e^{-2|k|^2t}|\widehat{\omega}_0(k)|^2=e^{-2t}\sum_{k\neq 0}e^{-2(|k|^2-1)t}|\widehat{\omega}_0(k)|^2\le e^{-2t}\|\omega_0\|_0^2 \tag{32}$$

Let $\omega_0\in El_\rho^\beta$, i.e. $\|\omega_0\|_{-\beta}\le\rho$. Formula (19) and inequalities (32), (23) imply (26) if $\alpha\ge 1/(1-c_1\rho)$. But the last inequality is equivalent to $\rho\le(\alpha-1)/(\alpha c_1)$.

This Lemma is analog of local existence theorem for 3D Navier-Stokes equations obtained in [4] with help of Fujita-Kato approach [5] and of local existence theorem for NPE connected with Burgers equation (see [3]). The proof of Lemma 4 is essentially easier then proofs of aforementioned results because here we use explicit formula (19) for solutions.

We show in this subsection that, actually, $M_-(\alpha)$ is essentially wider than El_ρ^β. For this goal we consider one infinite-dimensional subspace and show that it belongs to $M_-(\alpha)$

Let introduce the following subset U_L of $\mathbb{Z}^3\setminus\{0\}$:

$$U_L=\{\xi\in\mathbb{Z}^3\setminus\{0\}:\quad \xi+\eta-\zeta\neq 0\quad \forall\xi,\eta,\zeta\in U_L\} \tag{33}$$

An example of the subset belonging to U_L is the following set:

$$\{k=(k_1,k_2,k_3)\in\mathbb{Z}^3\setminus\{0\}: k_1+k_2+k_3\quad \text{is odd number }\}$$

Lemma 5. *The subspace*

$$L=\{\omega_0=\sum_{k\in U_L}(z_ke^{ik\cdot x}+\overline{z}_ke^{-ik\cdot x}),\quad z_k\in\mathbb{C}^3,\quad z_k\cdot k=0\}\subset V^0(\mathbb{T}^3) \tag{34}$$

belongs to $M_-(\alpha)$ if U_L is the set (33). Moreover

$$\forall\omega_0\in L\quad \Phi(S(t,\cdot,\omega_0))\equiv 0\quad \forall t\ge 0 \tag{35}$$

The proof of this Lemma is similar to analogous Lemma from [3].

Lemmas 4, 5 imply that $M_-(\alpha)\neq\emptyset$.

4.3 Certain Sets of Unit Sphere of V^0

Let denote the unit sphere of the phase space V^0 as follows:

$$\Sigma=\{v\in V^0:\quad \|v\|_0=1\} \tag{36}$$

To understand better the structure of phase flow corresponding to problem (17),(8) we introduce on Σ several sets. Define

$$A_-(t)=\{v\in\Sigma:\int_0^t\Phi(S(\tau,\cdot,v))d\tau\le 0\},$$
$$A_+(t)=\{v\in\Sigma:\int_0^t\Phi(S(\tau,\cdot,v))d\tau\ge 0\},$$
$$A_0(t)=\{v\in\Sigma:\int_0^t\Phi(S(\tau,\cdot,v))d\tau= 0\},$$

and

$$A_- = \cap_{t\geq 0} A_-(t), \quad A_+ = \cap_{t\geq 0} A_+(t), \quad A_0 = \cap_{t\geq 0} A_0(t) \tag{37}$$

All these sets are closed and nonempty. For instance, $A_0 \neq \emptyset$ in virtue of Lemma 5. Sets $A_\pm(t), A_\pm$ possess nonempty interior in topology of Σ, i.e. in the topology induced on Σ by topology of the space V^0. This assertion follows from continuity of the functional $V^{-\beta} \ni v \to \int_0^t \Phi(S(\tau,\cdot,v))d\tau$ with $\beta < 1/2$ and in particular for $\beta = 0$ (see Remark 1). Evidently, $A_0 = A_- \cap A_+$.

Linearity on v of operator $S(t,x,v)$ and oddness of $\Phi(v)$ with respect to v imply

Lemma 6.

$$v \in A_- \quad \textit{if and only if} \quad -v \in A_+$$

Introduce also the sets

$$B_+ = \Sigma \setminus A_- \equiv \{v \in \Sigma :\ \exists t_0 > 0 \int_0^{t_0} \Phi(S(\tau,\cdot,v))d\tau > 0\}, \quad B_- = \Sigma \setminus A_+ \tag{38}$$

It is easy to see that the set B_+ is open in topology of Σ. Moreover, the boundary ∂B_+ of set B_+ is defined by the relation

$$\partial B_+ = \{v \in \Sigma :\ \forall t > 0 \int_0^t \Phi(S(\tau,\cdot,v))d\tau \leq 0,\ \exists t_0 > 0 :\ \int_0^{t_0} \Phi(S(\tau,\cdot,v))d\tau = 0\}$$

It is clear that $A_0 \subset \partial B_+$ and $\partial B_+ \setminus A_0 \neq \emptyset$.

4.4 On Structure of Phase Space V^0

Let us introduce the following function defined on the set B_+ of sphere Σ:

$$B_+ \ni v \to b(v) = \max_{t\geq 0} \int_0^t \Phi(S(\tau,v))d\tau \tag{39}$$

Evidently, $b(v) > 0$ and $b(v) \to 0$ as $v \to \partial B_+$. Let $\rho \in (0,1]$. We define the following map $\Gamma_\rho(v)$ that plays the key role in description of structure of phase flow generated by boundary value problem (17),(8):

$$B_+ \ni v \to \Gamma_\rho(v) = \frac{\rho}{b(v)} v \in V^0 \tag{40}$$

where $b(v)$ is function (39). Note that $\|\Gamma_\rho(v)\|_0 \to \infty$ as $v \to \partial B_+$.

Theorem 1. *Let $\alpha > 1$ be a parameter from definition of the stability set $M_-(\alpha)$ and $\rho = (\alpha-1)/\alpha$. Then the image $\Gamma_\rho(v)$, $v \in B_+$ of the map Γ_ρ divides the space V^0 on two separate parts. The part containing origin coincides with the set of stability $M_-(\alpha)$. The part of V^0 between $\Gamma_\rho(v)$, $v \in B_+$ and $\Gamma_1(v)$, $v \in B_+$ coincides with intermediate space $M_I(\alpha)$, and the rest part of V^0 coincides with the set of explosions M_+.*

The proof of this theorem will be given in some other place.

Theorem 1 implies that $M_+ \neq \emptyset$ and $M_I(\alpha) \neq \emptyset$.

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