Note on Level Set Functions

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Abstract. In this note a concept of ε-level set function is introduced, i.e. a function which approximates a level set function satisfying the Hamilton-Jacobi inequality. We prove that each Lipschitz continuous solution of the Hamilton-Jacobi inequality is an ε-level set function. Next, a numerical approximation of the level set function is presented, i.e. method for the construction of an ε-level set function.

Keywords: level set function, numerical approximation, shape optimization

1 Introduction

The goal of shape optimization is to deform and modify the admissible shapes in order to comply with a given cost function that needs to be optimized.

Let $D \subset R^n$ be a given bounded domain and $\Omega_t \subset D$, be a sets from a family of admissible shapes Θ, indexed by t from some set of indexes. Assume that a certain functional $J(\cdot)$ reaches its minimum value on the set Ω_t for a x_t^{min} function.

Consider the following shape optimization problem: find a set $\Omega_{opt} \in \Theta$, for which there exists a function x_{opt}^{min} such that the following formula holds

$$J_{\Omega_{opt}}(x_{opt}^{min}) \leq J_{\Omega_t}(x_t^{min}) \qquad \Omega_t \in \Theta,$$

that is

$$J(\Omega_{opt}) = \inf_{\Omega \in \Theta} J(\Omega).$$

Problem formulated this way is difficult to solve – the crucial part is the construction of the family Θ so a known mathematical methods could be used. While solving this problem we were inspired by the approach we found in paper [1], where minimization over a family of sets is turned into a minimization over functions. Following this idea e.g. the level set function could be used to connect sets with functions – it allows us to manipulate boundary of the given shape through the level set function. A very brief sketch of this approach (transformation from optimization over domains into optimization over functions) is given in section 2.1. Notice that whenever a computation is mentioned, it means that, due to numerical computations limits, we are able to find only an approximate solution to a given problem. This is why, in practice, when solving shape optimization problem with the help of level set functions, only an approximation of it could be used and this is the main aim of this paper: to present a numerical approximation of the level set function, i.e. we want to present a method for the construction of an ε-level set function.

2 Level set method

Let Ω be an open and connected subset of D for which there exists a continuous function $\Psi(x) : D \to R$ such that $\Omega = \{x \in D : \Psi(x) < 0\}$. In consequence, the boundary Γ of Ω is a set of all points $x \in D$, such that $\Psi(x) = 0$. Let $\phi : (t, x) \in [0,1] \times D \to R$ be any function of class C^1, such that

$$\phi(0, x) = \Psi(x), \quad x \in D.$$

If Ω is a subject to changes in time we can describe Ω and its boundary Γ at time t (denoted as Ω_t and Γ_t) as

$$\Omega_t(\phi) = \{x \in D : \phi(t, x) < 0\}$$

and

$$\Gamma_t(\phi) = \{x \in D : \phi(t, x) = 0\}.$$

Let $x : [0,1] \times \Gamma(0) \to D$ be a continuous function, which for every point $x_0 \in \Gamma(0)$ assigns its location at time t, $t \in [0,1]$, i.e. $x(t, x_0) = x \in \Gamma(t)$. Function $x(\cdot, x_0)$ represents the location of the point x_0 at successive time steps t, determining a trajectory starting from the point $x_0 \in \Gamma_0$. For fixed starting point x_0, a trajectory represents the movement of this point. Taking all points $x_0 \in \Gamma_0$ into account we have the movement of a given boundary of Ω. This is why we call a trajectory starting at x_0 a *deformation of the point* x_0. We call the family of trajectories for all points $x_0 \in \Gamma_0$ the *deformation of the initial domain* Ω.

Let $V_n(x(t, x_0))$, $t \in [0,1]$, $x_0 \in \Gamma_0(\phi)$ be a Lipschitz mapping assigning to every point $x(y, x_0)$ its speed of movement in a normal direction to the boundary $\Gamma_t(\phi)$. A well known level set formula (e.g. [4]) according to which the changes of the function $\phi(t, \cdot)$ affect the boundary Γ_t takes the following form

$$\frac{\partial \phi}{\partial t}(t, x(t, x_0)) + |\nabla \phi(t, x(t, x_0))|\, V_n(x(t, x_0)) = 0 \tag{1}$$

Thus ϕ has to satisfy the following equation of Hamilton-Jacobi type

$$\frac{\partial \phi}{\partial t}(t, x) + |\nabla \phi(t, x)|\, V_n(x) = 0, \qquad (t, x) \in (0,1) \times D$$

with initial condition

$$\phi(0, x) = \Psi(x), \quad x \in D. \tag{2}$$

2.1 Problem reformulation

Denote by

$$\digamma = \{\Psi :\ \Psi \in C(\bar{\Omega}) \text{ and } \Psi = 0 \text{ on } \partial\Omega_\Psi,\ \bar{\Omega}_\Psi \subset \bar{\Omega},\ \partial\Omega_\Psi\text{-smooth}\}$$

Put $\Omega_t(\phi_t) = (x \in \Omega \mid \phi_t < 0)$. The family Θ of sets over which our shape optimization problem is considered can be defined as

$$\Theta = \{\Omega_t(\phi_t) : t \in [0,1],\ \phi_t = \phi(t,\cdot),\ \phi \in Lips([0,1], \bar{\Omega}_\Psi)\}$$

where ϕ satisfies (1) in Ω_Ψ, $\Psi \in \digamma$, with boundary condition (2) on $\partial\Omega_\Psi$, $\Psi \in \digamma$. Define a new family

$$\Phi = \{\phi_t : \phi_t = \phi(t,\cdot),\ \Omega_t(\phi_t) \in \Theta,\ t \in [0,1]\}$$

Now we can reformulate the shape optimization problem to the following problem

$$J(\Omega_{opt}) = \inf_{\phi_t \in \Phi} J(\phi_t).$$

2.2 ε - level set function

However, from the practical point of view only an approximate solution to (1) is considered, i.e. a solution $\phi_\varepsilon(\cdot,\cdot)$, which instead of an equality satisfies an inequality

$$-\varepsilon \le \frac{\partial \phi_\varepsilon}{\partial t}(t, x(t,x_0)) + |\nabla \phi_\varepsilon(t, x(t,x_0))|\, V_n(x(t,x_0)) \le 0. \qquad (3)$$

Therefore, instead of a level set function we have its approximation. We call a function $(t,x) \to \phi_\varepsilon(t,x)$, defined in $[0,1] \times \Omega$, an $\varepsilon-$ level set function if

$$-\varepsilon \le \phi_\varepsilon(t, x(t,x_0)) \le 0, \quad (t,x_0) \in [0,1] \times \partial\Omega, \qquad (4)$$

$$\Psi(x) \le \phi_\varepsilon(0,x) \le \Psi(x) + \varepsilon/2, \quad x \in \bar{\Omega}. \qquad (5)$$

It is also well known that there exists a Lipschitz continuous $\varepsilon-$ level set function and that it satisfies the Hamilton - Jacobi inequality

$$-\varepsilon \le \frac{\partial \phi_\varepsilon}{\partial t}(t,x) + |\nabla \phi_\varepsilon(t,x)|\, V_n(t,x) \le 0 \qquad (6)$$

and initial condition (5). We have the following theorem, which is very important from the numerical point of view.

Theorem 1. *Each element of the set W_ε,*

$$W_\varepsilon = \Big\{ w(t,x) \text{ is Lipschitz: } -\frac{\varepsilon}{2} \le w(0,x) \le 0,\ x \in \partial\Omega;$$
$$-\frac{\varepsilon}{2} \le \frac{\partial}{\partial t} w(t,x) + |\nabla w(t,x)|\, V_n(t,x) \le 0,\ a.a.\ (t,x) \in [0,1] \times \Omega \Big\}$$

is an $\varepsilon-$ level set function, i.e. it satisfies (4)-(5).

Proof. We use the ideas of a proof from [2]. Let t_0, $0 < t_0 \leq 1$ and $\delta > 0$, be such that the interval $[\delta, t_0 - \delta]$ is nonempty; let x_0, $x_0 \in \partial\Omega$, be an arbitrary initial value and let the $x(t), t \in [0, t_0 - \delta]$ start at x_0. Of, course, by assumptions the values of $x(t)$ are then bounded on $[\delta, t_0 - \delta]$, i.e. there is some compact set Q such that $x(t) \in Q$, $t \in [\delta, t_0 - \delta]$. Let $B_\tau(R^n)$ be a ball in R^n (with Euclidean norm) with radius $\tau \in R$ and center at 0. Denote $Q_1 = Q + B_1(R^n)$. Take $w \in W_\varepsilon$ and $\alpha \in R$ such that $0 < \alpha < \varepsilon/4$ and define

$$w_1(t,x) = w(t,x) + \alpha(t-1), \;\; (t,x) \in [0,1] \times \Omega.$$

Then, for a.a. $(t,x) \in [0,1] \times \Omega$, w_1 satisfies

$$\alpha - \frac{\varepsilon}{2} \leq \frac{\partial}{\partial t} w_1(t,x) + |\nabla w_1(t,x)| \, V_n(t,x)\} \leq \alpha.$$

Let us choose $0 < \beta_0 < \min\{1, \delta\}$ and define a function $(t,x) \to w_2^{\beta_0}(t,x)$ on $[\delta, t_0 - \delta] \times Q$ by the convolution $w_2^{\beta_0}(t,x) = (w_1 * \rho_{\beta_0})(t,x)$ where

$$\rho_{\beta_0}(t,x) = \frac{1}{\beta_0^{n+1}} \rho_1(\frac{t}{\beta_0}, \frac{x}{\beta_0}),$$

$$\int_{R^{n+1}} \rho_1(t,x) dt dx = 1, \;\; \text{supp } \rho_1 \subset B_1(R^{n+1}).$$

We claim that there exists $\beta' > 0$ such that for $\beta \leq \beta'$ and $(t,x) \in [\delta, t_0 - \delta] \times Q$,

$$\frac{1}{2}\alpha - \frac{\varepsilon}{2} \leq \frac{\partial}{\partial t} w_2^\beta(t,x) + \left|\nabla w_2^\beta(t,x)\right| V_n(t,x) \leq \frac{3}{2}\alpha. \tag{7}$$

Indeed, since $w_1(t,x)$ is Lipschitz continuous, there exists M, such that $\left|\frac{\partial}{\partial x} w_1\right| \leq M$ and

$$\begin{aligned} &\left|\left|\nabla w_2^\beta(t,x)\right| V_n(t,x) - \left(\left(|\nabla w_1(\cdot,\cdot)| \, V_n(\cdot,\cdot)\right) * \rho_\beta\right)(t,x)\right| \\ &\leq \int_{B_\beta(R^{n+1})} |\nabla w_1(t-s, x-y)| \, |V_n(t,x) - V_n(t-s, x-y, u)| \, \rho_\beta(s,y) ds dy \\ &\leq M \sup_{\substack{(t,x) \in [\delta, t_0 - \delta] \times Q \\ (s,y) \in B_\beta(R^{n+1})}} |V_n(t,x) - V_n(t-s, x-y)| \, . \end{aligned}$$

The right-hand side of the inequality presented above tends to zero as $\beta \to 0$ and that on $[\delta, t_0 - \delta] \times Q$, there is $\beta_2 > 0$ such that for $\beta \leq \beta_2$,

$$\left|\left|\nabla w_2^\beta(t,x)\right| V_n(t,x) - \left(\left(|\nabla w_1(\cdot,\cdot)| \, V_n(\cdot,\cdot)\right) * \rho_\beta\right)(t,x)\right| < \frac{\alpha}{2}.$$

Let us put on $[\delta, t_0 - \delta] \times Q$,

$$\begin{aligned} F(t,x) &= \frac{\partial}{\partial t} w_2^\beta(t,x) + \left|\nabla w_2^\beta(t,x)\right| V_n(t,x) \\ &= \left(\left(\frac{\partial w_1}{\partial t}(\cdot,\cdot) + |\nabla w_1(\cdot,\cdot)| \, V_n(\cdot,\cdot)\right) * \rho_\beta\right)(t,x) \\ &+ \left|\nabla w_2^\beta(t,x)\right| V_n(t,x) - \left(\left(|\nabla w_1(\cdot,\cdot)| \, V_n(\cdot,\cdot)\right) * \rho_\beta\right)(t,x). \end{aligned}$$

Considering the above estimations, we can find $0 < \beta' \leq \min\{\beta_0, \beta_1, \beta_2\}$ such that for $\beta \leq \beta'$,

$$\frac{1}{2}\alpha - \frac{\varepsilon}{2} \leq \left(\left(\alpha - \frac{\varepsilon}{2}\right) * \rho_\beta\right)(t,x) - \frac{\alpha}{2} \leq F(t,x)$$
$$\leq (\alpha * \rho_\beta)(t,x) + \frac{\alpha}{2} = \frac{3}{2}\alpha, \text{ for } (t,x) \in [\delta, t_0 - \delta] \times Q.$$

It is clear that $w_2^\beta(\cdot,\cdot)$ is $C^\infty([\delta, t_0-\delta] \times Q)$ and the function $(t,x) \to F(t,x)$, is continuous on $[\delta, t_0 - \delta] \times Q$. After integrating the inequalities (7) in $[\delta, t_0 - \delta]$ and considering the definition of V_n,

$$\begin{aligned} &\int_\delta^{t_0-\delta} \left(\frac{1}{2}\alpha - \frac{\varepsilon}{2}\right) dt \\ &\leq \int_\delta^{t_0-\delta} \left(\frac{\partial}{\partial t} w_2^\beta(t,x) + \left\{\left|\nabla w_2^\beta(t,x)\right| V_n(t,x)\right\}\right) dt \qquad (8) \\ &\leq \int_\delta^{t_0-\delta} \frac{3}{2}\alpha dt. \end{aligned}$$

As a consequence of (8), the following are obtained:

$$\left(\frac{1}{2}\alpha - \frac{\varepsilon}{2}\right)(t_0 - 2\delta) \leq \left(\int_\delta^{t_0-\delta} \frac{d}{dt} w_2^\beta(t, x(t)) dt\right) \leq \frac{3}{2}\alpha(t_0 - 2\delta)$$

and

$$\left(\frac{1}{2}\alpha - \frac{\varepsilon}{2}\right)(t_0 - 2\delta) \leq w_2^\beta(t_0 - \delta, x(t_0 - \delta)) \qquad (9)$$
$$-w_2^\beta(\delta, x(\delta)) \leq \frac{3}{2}\alpha(t_0 - 2\delta).$$

By the property of convolution, we see that $w_2^\beta \to w_1$ uniformly on $[\delta, t_0 - \delta] \times Q$ and thus (9) leads to

$$\left(\frac{1}{2}\alpha - \frac{\varepsilon}{2}\right)(t_0 - 2\delta) \leq w_1(t_0 - \delta, x(t_0 - \delta))$$
$$-w_1(\delta, x(\delta)) \leq \frac{3}{2}\alpha(t_0 - 2\delta).$$

Taking the limit with $\alpha \to 0$, we obtain

$$-\frac{\varepsilon}{2}(t_0 - 2\delta) \leq w(t_0 - \delta, x(t_0 - \delta)) - w(\delta, x(\delta)) \leq 0$$

Since δ was chosen arbitrarily and

$$-\frac{\varepsilon}{2} \leq w(0, x_0) \leq 0,$$

we infer further that

$$-\varepsilon \leq w(t_0, x(t_0) \leq 0.$$

Since t_0 and $w \in W_\varepsilon$ were chosen arbitrarily, the theorem is proved.

In this section, we proved that each Lipschitz continuous solution of the Hamilton-Jacobi inequality is an ε-level set function. As a direct conclusion of the theorem we infer the following corollary.

Corollary 1. *Let $\varepsilon_n > 0$, $\varepsilon_n \to 0$. Then each sequence of ε_n-level set functions $w_{\varepsilon_n} \in W_{\varepsilon_n}$ tends uniformly to the level set function $\phi(t,x)$ on $[0,1] \times \Omega$.*

3 Numerical approximation

In the second part of this paper we want to present a numerical approximation of the level set function for the equation (1), i.e. we want to present a method for the construction a ε-level set function for the equation (3), which satisfies (4) and (5). In order to achieve that, an adaptation of the method developed by J. Pustelnik in his Ph.D. thesis [3] is used.

Let $T \subset [0,1] \times D$ be a compact set and $(t,x) \to w(t,x)$ be a function defined on set T', $T \subset T'$ of class $C^2(T')$ such that

$$-\frac{\varepsilon}{2} \leq w(0,x) \leq 0,\ x \in \partial\Omega.$$

For $w(\cdot,\cdot)$ define now on the set T a new function $(t,x) \to F_w(t,x)$, corresponding to the left hand side of the formula (1)

$$F_w(t,x) := \frac{\partial w}{\partial t}(t,x) + |\nabla w(t,x)|\, V_n(x). \tag{10}$$

Function $(t,x) \to F_w(t,x)$ is a continuous function on T. Moreover it is also a Lipschitz function on T let M_{F^w} be a Lipschitz constant for the function $F_w(\cdot,\cdot)$. Owing to the compactness of T, function $F_w(\cdot,\cdot)$ reaches its lower and upper limits denoted respectively as k_l and k_u.

Let $\eta > 0$ be any fixed real number and $\{y_j^\eta\}_{j\in\mathbb{Z}}$ a sequence of numbers such that $y_0^\eta = 0$ and $y_{j+1}^\eta - y_j^\eta = \eta$ for $j \in \mathbb{Z}$. Define a new set J

$$J := \{j \in \mathbb{Z} :\ \exists_{(t,x)\in T}\ y_j^\eta < F_w(t,x) \leq y_{j+1}^\eta\}.$$

and let $P_T = \{P_j^{\eta,w}\}_{j\in J}$ be a family of sets covering the set T where

$$P_j^{\eta,w} := \left\{(t,x) \in T : y_j^\eta < F_w(t,x) \leq y_{j+1}^\eta\right\}$$

As a consequence of the definition of the family P_T and uniform continuity of the function $F_w(\cdot,\cdot)$ on the set T we have the following proposition

Proposition 1. *There exists a real number $\varepsilon > 0$, such that for every point $(t,x) \in T$ a ball with radius ε centered in (t,x) is covered either by one set $P_j^{\eta,w}$, $j \in J$ or by two sets $P_{j_1}^{\eta,w}$, $P_{j_2}^{\eta,w}$, $j_1, j_2 \in J$ and $|j_1 - j_2| = 1$.*

Let $h^{\eta,w}(\cdot,\cdot)$ be a function defined on T as follows

$$h^{\eta,w}(t,x) := -y_{j+1}^\eta \text{ for } (t,x) \in P_j^{\eta,w}, \quad j \in J. \tag{11}$$

As a consequence of the above definition we have

$$\underset{(t,x)\in T}{\forall} - \eta \leq F_w(t,x) + h^{\eta,w}(t,x) \leq 0. \tag{12}$$

Lemma 1. *Let $x_w(\cdot, x_0)$ be a deformation of any point x_0. There exists an increasing sequence of m points $\{t_i\}_{i=1,\dots,m}$, $t_1 = 0$ and $t_m = 1$ such that*

$$\underset{t\in[t_i,t_{i+1}]}{\forall} |F_w(t_i, x_w(t_i, x_0)) - F_w(t, x_w(t, x_0))| \le \frac{\eta}{2},\ i = 1, \dots, m-1. \tag{13}$$

Proof. This is a simple consequence of the absolute continuity of $x_w(\cdot, \cdot)$.

Notice, that Lemma 1 holds for any $\tau \in [0, 1]$, since for any $\tau \in [0, 1]$ there exists an increasing sequence of m_τ points $\{t_i^\tau\}_{i=1,\dots,m_\tau}$, where $t_1^\tau = 0$ and $t_{m_\tau}^\tau = \tau$, for which the following formula holds

$$\underset{t\in[t_i,t_{i+1}]}{\forall} |F_w(t_i, x_w(t_i, x_0)) - F_w(t, x_w(t, x_0))| \le \frac{\eta}{2},\ i = 1, \dots, m_\tau - 1.$$

Moreover, having the aforementioned sequence for $\tau = 1$, we can easily determine a sequence for any $\tau = \{t_i^1\}_{i=1,\dots,m_1}$. As a consequence of the formula (13) we have that for any $i \in \{1, \dots, m_\tau - 1\}$, if $(t_i, x_w(t_i, x_0)) \in P_j^{\eta,w}$ for some $j \in J$, than for every $x_0 \in \Gamma_0$ the following property holds

$$\underset{t\in[t_i,t_{i+1}]}{\forall} (t, x_w(t, x_0)) \in P_{j-1}^{\eta,w} \cup P_j^{\eta,w} \cup P_{j+1}^{\eta,w}.$$

From the above and Definition (11) for all $t \in [t_i, t_{i+1}]$ we get that

$$[h^{\eta,w}(t_i, x_w(t_i, x_0)) - \eta \le h^{\eta,w}(t, x_w(t, x_0)) \le h^{\eta,w}(t_i, x_w(t_i, x_0)) + \eta]\,. \tag{14}$$

Particularly for every $i \in \{2, \dots, m_\tau - 1\}$

$$h^{\eta,w}(t_i, x_w(t_i, x_0)) - h^{\eta,w}(t_{i-1}, x_w(t_{i-1}, x_0)) = \eta^i_{x_w(\cdot,x_0)}, \tag{15}$$

where $\eta^i_{x_w(\cdot,\cdot)} \in \{-\eta, 0, \eta\}$. Integration of (14) results, for any $i \in \{1, \dots, m_\tau - 1\}$, in the following double inequality

$$\begin{aligned} &[h^{\eta,w}(t_i, x_w(t_i, x_0)) - \eta]\,(t_{i+1} - t_i) \\ &\le \int_{t_i}^{t_{i+1}} h^{\eta,w}(t, x_w(t, x_0))dt \le [h^{\eta,w}(t_i, x_w(t_i, x_0)) + \eta]\,(t_{i+1} - t_i) \end{aligned}$$

and in consequence

$$\begin{aligned} &\sum_{i\in 1,\dots,m_\tau-1} [h^{\eta,w}(t_i, x_w(t_i, x_0))(t_{i+1} - t_i)] - \eta\tau \\ &\le \int_0^\tau h^{\eta,w}(t, x_w(t, x_0))dt \\ &\le \sum_{i\in 1,\dots,m_\tau-1} [h^{\eta,w}(t_i, x_w(t_i, x_0))(t_{i+1} - t_i)] + \eta\tau. \end{aligned} \tag{16}$$

Owing to the fact, that by simple calculation the expression

$$\sum_{i\in 1,\dots,m_\tau-1} [h^{\eta,w}(t_i, x_w(t_i, x_0))(t_{i+1} - t_i)]$$

can be substituted by some sum of differences (15), finally formula (16) takes the following form

$$\begin{aligned}&\sum_{i\in 2,\dots,m_\tau-1}\eta^i_{x_w(\cdot,x_0)}(\tau-t_i)+h^{\eta,w}(0,x_w(0,x_0))\tau-\eta\tau\\&\le\int_0^\tau h^{\eta,w}(t,x_w(t,x_0))dt\\&\le\sum_{i\in 2,\dots,m_\tau-1}\eta^i_{x_w(\cdot,x_0)}(\tau-t_i)+h^{\eta,w}(0,x_w(0,x_0))\tau+\eta\tau.\end{aligned}\tag{17}$$

Notice that inequality (17) is very useful in computation. It allows estimation of an integral of function $h^{\eta,w}(\cdot,\cdot)$ along deformation $x_w(\cdot,x_0)$ as a finite sum of values from the set $\{-\eta,0,\eta\}$. Moreover for any two deformations of two different points $x_0^1\in\Gamma_0$ and $x_0^2\in\Gamma_0$ values

$$\sum_{i\in 2,\dots,m_\tau-1}\eta^i_{x_w(\cdot,x_0^1)}(\tau-t_i)+h^{\eta,w}(0,x_w(0,x_0^1))\tau$$

and

$$\sum_{i\in 2,\dots,m_\tau-1}\eta^i_{x_w(\cdot,x_0^2)}(\tau-t_i)+h^{\eta,w}(0,x_w(0,x_0^2))\tau$$

are equal if the following conditions hold

$$\eta^i_{x_w(\cdot,x_0^1)}=\eta^i_{x_w(\cdot,x_0^2)}\text{ for every } i\in\{2,\dots,m_\tau-1\},\tag{18}$$

$$x_0^1\in P_j^{\eta,w}\text{ i }x_0^2\in P_j^{\eta,w}, j\in J.\tag{19}$$

In consequence, in the set K of all deformations $x_w(\cdot,x_0)$, $x_0\in\Gamma_0$ an equivalence relation E can be introduced, taking as an equivalent any two deformations $x(\cdot,x_0^1)$ and $x(\cdot,x_0^2)$, $x_0^1,x_0^2\in\Gamma_0$ fulfills (18) and (19). The cardinality of a set K_E of all disjoint equivalence class of relation E is finite and limited from above by value $3^{m_\tau-1}$. Now define a set X of $m_\tau-1$-dimensional vectors $x=(x_1,\dots,x_{m_\tau-1})$, where $x_1=0$ and $x_i=\eta^i_{x_w^j}$, $\quad i=2,\dots,m_\tau-1$, while $x_w^j\in X_E$ is any element of j-th equivalence class, $i=1,\dots,||K_E||$. Inequality (17) can be rewritten as

$$\begin{aligned}&\sum_{i\in 1,\dots,m_\tau-1}x^i_{x_w(\cdot,x_0)}(\tau-t_i)+h^{\eta,w}(0,x_w(0,x_0))\tau-\eta\tau\\&\le\int_0^\tau h^{\eta,w}(t,x_w(t,x_0))dt\\&\le\sum_{i\in 1,\dots,m_\tau-1}x^i_{x_w(\cdot,x_0)}(\tau-t_i)+h^{\eta,w}(0,x_w(0,x_0))\tau+\eta\tau.\end{aligned}\tag{20}$$

Thus infinite space of all deformation can be reduced to the finite set.

Lemma 2. *If x_0 is any point from Γ_0 and $\tau \in [0,1]$ then we have the following inequality*

$$\begin{aligned}
&-\sum_{i\in 2,\dots,m_\tau-1} x^i_{x_w(\cdot,x_0)}(\tau-t_i) - h^{\eta,w}(0,x_w(0,x_0))\tau - 2\eta\tau \\
&\le w(\tau,x(\tau)) - w(0,x_0) \\
&\le -\sum_{i\in 2,\dots,m_\tau-1} x^i_{x_w(\cdot,x_0)}(\tau-t_i) - h^{\eta,w}(0,x_w(0,x_0))\tau + \eta\tau.
\end{aligned}$$

Proof. Integration of (12) along any deformation $x(\cdot,x_0)$ on interval $[0,\tau]$ gives

$$-\eta\tau - \int_0^\tau h^{\eta,w}(t,x)dt \le \int_0^\tau F_w(t,x)dt \le -\int_0^\tau h^{\eta,w}(t,x)dt,$$

and in consequence

$$\begin{aligned}
&-\eta\tau - \int_0^\tau h^{\eta,w}(t,x)dt \\
&\le \int_0^\tau \left(\frac{\partial w}{\partial t}(t,x(t,x_0)) + |\nabla w(t,x(t,x_0))|\, V_n(t,x(t,x_0))\right) dt \\
&\le -\int_0^\tau h^{\eta,w}(t,x)dt.
\end{aligned}$$

Considering equation (17) we have

$$\begin{aligned}
&-\sum_{i\in 2,\dots,m_\tau-1} x^i_{x_w(\cdot,x_0)}(\tau-t_i) - h^{\eta,w}(0,x_w(0,x_0))\tau - 2\eta\tau \\
&\le \int_0^\tau \left(\frac{\partial w}{\partial t}(t,x(t,x_0)) + |\nabla w(t,x(t,x_0))|\, V_n(t,x(t,x_0))\right) dt \\
&\le -\sum_{i\in 2,\dots,m_\tau-1} x^i_{x_w(\cdot,x_0)}(\tau-t_i) - h^{\eta,w}(0,x_w(0,x_0))\tau + \eta\tau,
\end{aligned}$$

and finally because

$$\begin{aligned}
&\int_0^\tau \left(\frac{\partial w}{\partial t}(t,x(t,x_0)) + |\nabla w(t,x(t,x_0))|\, V_n(t,x(t,x_0))\right) dt \\
&= \int_0^\tau \frac{d}{dt} w(t,x(t,x_0))dt
\end{aligned}$$

we have

$$\begin{aligned}
&-\sum_{i\in 2,\dots,m_\tau-1} x^i_{x_w(\cdot,x_0)}(\tau-t_i) - h^{\eta,w}(0,x_w(0,x_0))\tau - 2\eta\tau \\
&\le w(\tau,x(\tau)) - w(0,x_0) \\
&\le -\sum_{i\in 2,\dots,m_\tau-1} x^i_{x_w(\cdot,x_0)}(\tau-t_i) - h^{\eta,w}(0,x_w(0,x_0))\tau + \eta\tau.
\end{aligned}$$

Theorem 2. *Set a real number $\eta > 0$, point $x_0 \in \partial\Omega$ and $\tau \in [0,1]$. Then*

$$\begin{aligned} &w(\tau, x(\tau)) - w(0, x_0) \\ &+ \sum_{i \in 2,\dots,m_\tau - 1} x^i_{x_w(\cdot, x_0)}(\tau - t_i) + h^{\eta,w}(0, x_w(0, x_0))\tau - \eta\tau \end{aligned}$$

is a value of some ε-level set function at the point $(\tau, x(\tau, x_0))$ for $\varepsilon = 3\eta\tau$.

Proof. From the Theorem 2 we obtain

$$\begin{aligned} -3\eta\tau &\le w(\tau, x(\tau)) - w(0, x_0) \\ &+ \sum_{i \in 2,\dots,m_\tau - 1} x^i_{x_w(\cdot, x_0)}(\tau - t_i) + h^{\eta,w}(0, x_w(0, x_0))\tau - \eta\tau \\ &\le 0. \end{aligned}$$

From the above we infer that it is enough to take into account a finite number of points from $\partial\Omega$ to get the approximation of the level set function with an error not greater than 3η.

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