Second Order Conditions for L^2 Local Optimality in PDE Control*

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Abstract. In the second order analysis of infinite dimension optimization problems, we have to deal with the so-called two-norm discrepancy. As a consequence of this fact, the second order optimality conditions usually imply local optimality in the L^∞ sense. However, we have observed that the L^2 local optimality can be proved for many control problems of partial differential equations. This can be deduced from the standard second order conditions. To this end, we make some quite realistic assumptions on the second derivative of the cost functional. These assumptions do not hold if the control does not appear explicitly in the cost functional. In this case, the optimal control is usually of bang-bang type. For this type of problems we also formulate some new second order optimality conditions that lead to the strict L^2 local optimality of the bang-bang controls.

Keywords: optimal control of partial differential equations, semilinear partial differential equations, second order optimality conditions, bang-bang controls

1 Introduction

This paper is split into three parts. In the first part, we consider the following infinite dimensional abstract optimization problem. Let U_∞ and U_2 be Banach and Hilbert spaces, respectively, endowed with the norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$. We assume that $U_\infty \subset U_2$ with continuous embedding; in particular, the choice $U_\infty = U_2$ is possible. A nonempty convex subset $\mathcal{K} \subset U_\infty$ is given, and $\mathcal{A} \subset U_\infty$ is an open set covering $\mathcal{K}$. Moreover, an objective function $J : \mathcal{A} \longrightarrow \mathbb{R}$ is given. We consider the abstract optimization problem

$$\text{(P)} \qquad \min_{u\in\mathcal{K}} J(u),$$

where we assume that J is of class C^2 with respect to the norm $\|\cdot\|_\infty$. In the next section, we will impose some other assumptions on J so that the first order

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optimality conditions and the inequality $J''(\bar{u})v^2 > 0$ for every $v \in C_{\bar{u}} \setminus \{0\}$ imply that $\bar{u}$ is a strict local minimum of (P) in the U_2 sense. Here $C_{\bar{u}}$ denotes the usual cone of critical directions that we will define later. This result is new in the sense that the classical theory claims the local optimality only in the U_∞ sense due to the non-differentiability of J in with respect to $\|\cdot\|_2$. Moreover, a stronger inequality $J''(\bar{u})v^2 \geq \delta\|v\|_2^2$ is usually required.

In the second part of the paper, contained in §3, we prove that the abstract assumptions are fulfilled by a typical Neumann control problem. The method used for this control problem can be extended in an easy form to many other control problems associated with elliptic or parabolic equations; see [8]. Finally, the last part of the paper is considered in §4. There, we analyze the case of bang-bang control problems, which do not satisfy the assumptions of §2. For these problems we also give some second order conditions leading to the strict L^2 local optimality of the controls.

2 An abstract optimization problem in Banach spaces

The results presented in this section were obtained in collaboration with Fredi Tröltzsch. The reader is referred to [8] for the proofs and details.

In this section, we study the abstract optimization problem (P) formulated in the introduction. Besides the hypotheses established in §1 on U_2 and U_∞, we require the following assumptions on (P).

(A1) The functional $J : \mathcal{A} \longrightarrow \mathbb{R}$ is of class C^2. Furthermore, for every $u \in \mathcal{K}$ there exist continuous extensions

$$J'(u) \in \mathcal{L}(U_2, \mathbb{R}) \quad \text{and} \quad J''(u) \in \mathcal{B}(U_2, \mathbb{R}), \tag{2.1}$$

where $\mathcal{L}(U_2, \mathbb{R})$ and $\mathcal{B}(U_2, \mathbb{R})$ denote the spaces of continuous linear and bilinear forms on U_2, respectively.

(A2) For any sequence $\{(u_k, v_k)\}_{k=1}^\infty \subset \mathcal{K} \times U_2$ with $\|u_k - \bar{u}\|_2 \to 0$ and $v_k \rightharpoonup v$ weakly in U_2, the conditions

$$J'(\bar{u})v = \lim_{k\to\infty} J'(u_k)v_k, \tag{2.2}$$

$$J''(\bar{u})v^2 \leq \liminf_{k\to\infty} J''(u_k)v_k^2, \tag{2.3}$$

$$\text{if } v = 0, \text{ then } \quad \Lambda \liminf_{k\to\infty} \|v_k\|_2^2 \leq \liminf_{k\to\infty} J''(u_k)v_k^2, \tag{2.4}$$

hold for some $\Lambda > 0$.

The reader might have the impression that Assumptions *(A1)* and *(A2)*, mainly *(A2)*, are too strong. However, we will see in the next sections that they are fulfilled by many optimal control problems.

Associated with $\bar{u}$, we define the sets

$$\begin{aligned} S_{\bar{u}} &= \{v \in U_\infty : v = \lambda(u - \bar{u}) \text{ for some } \lambda > 0 \text{ and } u \in \mathcal{K}\}, \\ C_{\bar{u}} &= cl_2(S_{\bar{u}}) \cap \{v \in U_2 : J'(\bar{u})v = 0\} \\ D_{\bar{u}} &= \{v \in S_{\bar{u}} : J'(\bar{u})v = 0\}, \end{aligned} \tag{2.5}$$

where $cl_2(S_{\bar u})$ denotes the closure of $S_{\bar u}$ in U_2. The set $S_{\bar u}$ is called the cone of feasible directions and $C_{\bar u}$ is said to be the critical cone. It is obvious that $cl_2(D_{\bar u}) \subset C_{\bar u}$. However, the equality can fail. In fact, this equality is a regularity condition equivalent to the notion of polyhedricity of $\mathcal{K}$; see [2] or [1, §3.2]. This property is enjoyed by control problems with pointwise control constraints.

Now, we formulate the necessary first and second order optimality conditions. The second order conditions hold under the mentioned regularity assumption; we refer to [1, §3.2] or [7] for the proof.

Theorem 2.1. *Assume that (A1) holds and let $\bar u$ be a local solution of* (P) *in U_∞, then $J'(\bar u)(u-\bar u) \geq 0$ $\forall u \in \mathcal{K}$. Moreover, if the regularity condition $C_{\bar u} = cl_2(D_{\bar u})$ is satisfied, then $J''(\bar u)v^2 \geq 0$ holds for all $v \in C_{\bar u}$.*

Now, we state our result about sufficient sufficient second order optimality conditions. As the reader may check, the gap between the necessary and sufficient second order conditions is minimal, the same as in finite dimension.

Theorem 2.2. *Suppose that assumptions (A1) and (A2) hold. Let $\bar u \in \mathcal{K}$ satisfy the first order optimality condition as formulated in Theorem 2.1, and*

$$J''(\bar u)v^2 > 0 \quad \forall v \in C_{\bar u} \setminus \{0\}. \tag{2.6}$$

Then, there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$J(\bar u) + \frac{\delta}{2}\|u-\bar u\|_2^2 \leq J(u) \quad \forall u \in \mathcal{K} \cap B_2(\bar u;\varepsilon). \tag{2.7}$$

Above $B_2(\bar u;\varepsilon)$ denotes the ball of U_2 with center at $\bar u$ and radius ε.

This theorem can be proved arguing by contradiction. To this end, we assume that for any positive integer k there exists $u_k \in \mathcal{K}$ such that

$$\|u_k-\bar u\|_2 < \frac{1}{k} \quad \text{and} \quad J(\bar u) + \frac{1}{2k}\|u_k-\bar u\|_2^2 > J(u_k). \tag{2.8}$$

Setting $\rho_k = \|u_k - \bar u\|_2$ and $v_k = (u_k - \bar u)/\rho_k$, we can assume that $v_k \rightharpoonup v$ in U_2; if necessary, we select a subsequence. Then we prove that $v \in C_{\bar u}$ and $J''(\bar u)v^2 = 0$. Because of (2.6), this is only possible if $v = 0$. With the help of (2.4) the contradiction is obtained from the identity $\|v_k\|_2 = 1$; see [8] for the details.

As a consequence of Theorem 2.2, we can not only prove that $\bar u$ is the unique local minimum in a certain U_2 neighborhood. We are even able to show the non-existence of other stationary points in such a neighborhood. Recall that $\tilde u \in \mathcal{K}$ is said to be a stationary point if

$$J'(\tilde u)(u-\tilde u) \geq 0 \quad \text{for all} \quad u \in \mathcal{K}. \tag{2.9}$$

Corollary 3. *Under the assumptions of Theorem 2.2, there exists $\varepsilon > 0$ such that there is no stationary point $\tilde{u} \in B_2(\bar{u};\varepsilon) \cap \mathcal{K}$ different from $\bar{u}$.*

Assumption (2.6) has another consequence that was known up to now only in an U_∞-neighborhood of $\bar{u}$. The result expresses some alternative formulation of second-order sufficient conditions that is useful for applications in the numerical analysis.

Theorem 2.4. *Under the assumptions of Theorem 2.2, there exist numbers $\varepsilon > 0$, $\nu > 0$ and $\tau > 0$ such that*

$$J''(u)v^2 \geq \frac{\nu}{2}\|v\|_2^2 \quad \forall\, v \in E^\tau_{\bar{u}} \quad \textit{and} \quad \forall\, u \in \mathcal{K} \cap B_2(\bar{u};\varepsilon), \tag{2.10}$$

where $$E^\tau_{\bar{u}} = \{v \in cl_2(S_{\bar{u}}) : |J'(\bar{u})v| \leq \tau\|v\|_2\}.$$

3 Application. An Elliptic Neumann Control Problem

In this section we study the optimal control problem

$$(\mathrm{P}_1) \qquad \min_{u \in \mathcal{K}} J(u),$$

where $$J(u) = \int_\Omega L(x, y_u(x))\,dx + \int_\Gamma l(x, y_u(x), u(x))\,d\sigma(x), \tag{3.1}$$

$$\mathcal{K} = \{u \in L^\infty(\Gamma) : \alpha \leq u(x) \leq \beta \ \text{ for a.a. } x \in \Gamma\},$$

σ denotes the Lebesgue surface measure, $-\infty < \alpha < \beta < +\infty$, and y_u is the solution of the following Neumann problem

$$\begin{cases} -\Delta y + f(y) = 0 & \text{in } \Omega, \\ \partial_\nu y = u & \text{on } \Gamma. \end{cases} \tag{3.2}$$

We impose the following assumptions on the functions and parameters appearing in the control problem (P_1).
Assumption (N1): Ω is an open, bounded and connected subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary Γ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function of class C^2 such that $f'(t) \geq c_o > 0$ for all $t \in \mathbb{R}$. The reader is referred to [5] for more general non-linear terms in the state equation.
Assumption (N2): We assume that $L : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ and $l : \Gamma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions of class C^2 with respect to the second variable for L and with respect to the second and third variables for l, with $L(\cdot, 0) \in L^1(\Omega)$, $l(\cdot, 0, 0) \in L^1(\Gamma)$. For every $M > 0$ there exist functions $\psi_M \in L^{\bar{p}}(\Omega)$, $\bar{p} > n/2$,

and $\phi_M \in L^{\bar q}(\Gamma)$, $\bar q > n-1$, and a constant $C_M > 0$ such that

$$\begin{cases} \left|\dfrac{\partial^j L}{\partial y^j}(x,y)\right| \le \psi_M(x), \quad \text{with } j = 1,2, \\ \left|\dfrac{\partial^j l}{\partial y^j}(x,y,u)\right| \le \phi_M(x), \quad \text{with } j = 1,2, \\ \left|\dfrac{\partial^{i+j} l}{\partial u^i \partial y^j}(x,y,u)\right| \le C_M, \ \ 1 \le i+j \le 2 \text{ and } i \ge 1 \end{cases}$$

are satisfied for a.a. $x \in \Omega$ and every $u, y \in \mathbb{R}$, with $|y| \le M$ and $|u| \le M$.

Moreover, for every $\varepsilon > 0$ there exists $\eta > 0$ such that for a.a. $x \in \Omega$ and all $u_i, y_i \in \mathbb{R}$, with $i = 1,2$,

$$\begin{cases} |y_2 - y_1| \le \eta \Rightarrow \left|\dfrac{\partial^2 L}{\partial y^2}(x,y_2) - \dfrac{\partial^2 L}{\partial y^2}(x,y_1)\right| \le \varepsilon, \\ |u_2 - u_1| + |y_2 - y_1| \le \eta \Rightarrow \left|D^2_{(y,u)} l(x,y_2,u_2) - D^2_{(y,u)} l(x,y_1,u_1)\right| \le \varepsilon. \end{cases}$$

Here $D^2_{(y,u)} l(x,y,u)$ denotes the Hessian matrix of l with respect to the variables (y,u). We also assume the Legendre-Clebsch type condition

$$\exists \Lambda > 0 \text{ such that } \frac{\partial^2 l}{\partial u^2}(x,y,u) \ge \Lambda \ \text{ for a.a. } x \in \Gamma \text{ and } \forall y, u \in \mathbb{R}. \tag{3.3}$$

It is obvious that the usual quadratic integrands $L(x,y) = \frac{1}{2}(y - y_{Ld}(x))^2$ and $l(x,y,u) = \frac{1}{2}(y - y_{ld}(x))^2 + \frac{\Lambda}{2}u^2$ satisfy Assumption *(N2)* if $y_{Ld} \in L^{\bar p}(\Omega)$ and $y_{ld} \in L^{\bar q}(\Gamma)$.

The hypothesis (3.3) is crucial for satisfying the assumptions (2.3) and (2.4). In §4 we will consider the case where (3.3) doe not hold.

On the state equation (2.1), the following result is known.

Theorem 3.1. *Under the Assumption (N1), for every $u \in L^{\bar q}(\Gamma)$ the equation (3.2) has a unique solution $y_u \in H^1(\Omega) \cap C(\bar\Omega)$. Furthermore, the mapping $G : L^{\bar q}(\Gamma) \longrightarrow H^1(\Omega) \cap C(\bar\Omega)$, defined by $G(u) = y_u$, is of class C^2. For elements u, v, v_1 and v_2 of $L^{\bar q}(\Gamma)$, the functions $z_v = G'(u)v$ and $z_{v_1 v_2} = G''(u)(v_1, v_2)$ are the solutions of the problems*

$$\begin{cases} Az + f'(y_u)z = 0 & \text{in } \Omega, \\ \partial_{\nu_A} z = v & \text{on } \Gamma, \end{cases} \tag{3.4}$$

and

$$\begin{cases} Az + f'(y_u)z + f''(y_u)z_{v_1} z_{v_2} = 0 & \text{in } \Omega, \\ \partial_{\nu_A} z = 0 & \text{on } \Gamma, \end{cases} \tag{3.5}$$

respectively, where $z_{v_i} = G'(u)v_i$, $i = 1,2$.

The proof of existence and uniqueness of a solution y_u in $H^1(\Omega)\cap L^\infty(\Omega)$ is standard; see, for instance, [3]. For the continuity of y_u, the reader is referred to [11] or [12]. As usual, the differentiability of G can be obtained from the implicit function theorem.

As a consequence of this theorem and the chain rule the next result follows.

Theorem 3.2. *Assuming (N1) and (N2), then the mapping $J : L^\infty(\Gamma) \longrightarrow \mathbb{R}$, defined by (3.1), is of class C^2. For all u, v, v_1 and v_2 of $L^\infty(\Gamma)$ we have*

$$J'(u)v = \int_\Gamma \left(\varphi_u + \frac{\partial l}{\partial u}(x, y_u, u)\right) v\, d\sigma \tag{3.6}$$

$$\begin{aligned} J''(u)(v_1, v_2) &= \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x, y_u) - \varphi_u f''(y_u)\right) z_{v_1} z_{v_2}\, dx \\ &+ \int_\Gamma \left(\frac{\partial^2 l}{\partial y^2}(x, y_u, u) z_{v_1} z_{v_2} + \frac{\partial^2 l}{\partial y \partial u}(x, y_u, u)(v_1 z_{v_2} + v_2 z_{v_1})\right) d\sigma \\ &+ \int_\Gamma \frac{\partial^2 l}{\partial u^2}(x, y_u, u) v_1 v_2\, d\sigma, \end{aligned} \tag{3.7}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$, and $\varphi_u \in H^1(\Omega) \cap C(\bar\Omega)$ is the solution of

$$\left\{\begin{aligned} -\Delta\varphi + f'(y_u)\varphi &= \frac{\partial L}{\partial y}(x, y_u) && \text{in } \Omega, \\ \partial_\nu \varphi &= \frac{\partial l}{\partial y}(x, y_u, u) && \text{on } \Gamma. \end{aligned}\right. \tag{3.8}$$

From the above expressions for $J'(u)$ and $J''(u)$ and Assumption *(N2)* we deduce that $J'(u)$ and $J''(u)$ can be extended to linear and bilinear forms, respectively, on $L^2(\Gamma)$. Even more, there exist two constants $M_1 > 0$ and $M_2 > 0$ such that for every $v, v_1, v_2 \in L^2(\Gamma)$ and $u \in \mathcal{K}$

$$|J'(u)v| \le M_1 \|v\|_{L^2(\Gamma)} \quad \text{and} \quad |J''(u)(v_1, v_2)| \le M_2 \|v_1\|_{L^2(\Gamma)} \|v_2\|_{L^2(\Gamma)}. \tag{3.9}$$

This shows that (2.1) holds with $U_2 = L^2(\Gamma)$ and $U_\infty = L^\infty(\Gamma)$. The most delicate issue in the proof of (2.2)-(2.4) is the verification of (2.3), which can be done with the help of the following lemma.

Lemma 3.1 *Let (X, Σ, μ) be a measure space with $\mu(X) < +\infty$. Suppose that $\{g_k\}_{k=1}^\infty \subset L^\infty(X)$ and $\{v_k\}_{k=1}^\infty \subset L^2(X)$ satisfy the assumptions*

- *$g_k \ge 0$ a.e. in X, $\{g_k\}_{k=1}^\infty$ is bounded in $L^\infty(X)$ and $g_k \to g$ in $L^1(X)$.*
- *$v_k \rightharpoonup v$ in $L^2(X)$.*

Then there holds the inequality

$$\int_X g(x) v^2(x)\, d\mu(x) \le \liminf_{k\to\infty} \int_X g_k(x) v_k^2(x)\, d\mu(x). \tag{3.10}$$

The proof of this lemma can be obtained by an application of Egorov's theorem; see [8]. To confirm (2.3) we apply Lemma 3.1 with $X = \Gamma$, $\mu = \sigma$ and

$$0 < \Lambda \le g_k(x) = \frac{\partial^2 l}{\partial u^2}(x, y_{u_k}(x), u_k(x)) \to g(x) = \frac{\partial^2 l}{\partial u^2}(x, y_u(x), u(x)) \ \text{ in } L^1(\Gamma).$$

Finally, we apply Theorems 2.1 and 2.2 to the problem (P_1). Given $\bar u \in \mathcal{K}$, we see that the cone of critical directions $C_{\bar u}$ defined in §2 can be expressed for the problem (P_1) in the form

$$C_{\bar u} = \{v \in L^2(\Gamma) : v(x) = \begin{cases} \ge 0 \text{ if } \bar u(x) = \alpha \\ \le 0 \text{ if } \bar u(x) = \beta \\ \ \ 0 \ \text{ if } \bar d(x) \ne 0 \end{cases} \text{a.e. in } \Gamma\},$$

where

$$\bar d(x) = \bar\varphi(x) + \frac{\partial l}{\partial u}(x, \bar y(x), \bar u(x))$$

and $\bar y = y_{\bar u}$ and $\bar\varphi = \varphi_{\bar u}$ denote the state and adjoint state associated to $\bar u$, respectively. It is not difficult to check that the regularity assumption stated in Theorem 2.1 is fulfilled by $C_{\bar u}$. Then we have the following corollaries.

Corollary 3.1 *Let the Assumption (N1) be satisfied and suppose that $\bar u$ is a local minimum of (P_1) in the $L^\infty(\Gamma)$ sense. Then there holds $J'(\bar u)(u - \bar u) \ge 0$ for all $u \in \mathcal{K}$ and $J''(\bar u)v^2 \ge 0$ $\forall v \in C_{\bar u}$. Conversely, if $\bar u \in \mathcal{K}$ obeys*

$$J'(\bar u)(u - \bar u) \ge 0 \quad \forall u \in \mathcal{K}, \tag{3.11}$$

$$J''(\bar u)v^2 > 0 \quad \forall v \in C_{\bar u} \setminus \{0\}, \tag{3.12}$$

then there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$J(\bar u) + \frac{\delta}{2}\|u - \bar u\|^2_{L^2(\Gamma)} \le J(u) \quad \forall u \in \mathcal{K} \cap B_2(\bar u; \varepsilon). \tag{3.13}$$

Let us underline that the mapping G is only differentiable in $L^q(\Gamma)$ for $q > n-1$. Consequently, for all $n \ge 3$, G is not differentiable in $L^2(\Gamma)$. Moreover, the general nonlinear cost functional J is only differentiable in $L^\infty(\Gamma)$. Hence, for any dimension n, the classical theory of second order conditions would only assure the local optimality of $\bar u$ in the $L^\infty(\Gamma)$ sense. In contrast to this, our result guarantees local optimality in the sense of $L^2(\Gamma)$.

Corollary 3.2 *Under the assumption (N1) and (N2), there exists a ball $B_2(\bar u; \varepsilon)$ in $L^2(\Gamma)$ such that there is no other stationary point in $B_2(\bar u; \varepsilon) \cap \mathcal{K}$ than $\bar u$. Moreover, there exist numbers $\nu > 0$ and $\tau > 0$ such that*

$$J''(u)v^2 \ge \frac{\nu}{2}\|v\|^2_{L^2(\Gamma)} \quad \forall\, v \in C^\tau_{\bar u} \quad \text{and} \quad \forall\, u \in \mathcal{A} \cap B_2(\bar u; \varepsilon), \tag{3.14}$$

where $\mathcal{A}$ is a bounded open subset of $L^\infty(\Gamma)$ containing $\mathcal{K}$ and

$$C_{\bar u}^\tau = \{v \in L^2(\Gamma) : v(x) = \begin{cases} \geq 0 \text{ if } \bar u(x) = \alpha \\ \leq 0 \text{ if } \bar u(x) = \beta \\ \ 0 \quad \text{if } |\bar d(x)| > \tau \end{cases} \text{ a.e. in } \Gamma\}.$$

In the above corollaries, $B_2(\bar u;\varepsilon)$ denotes the $L^2(\Gamma)$-ball of radius ε centered at $\bar u$.

Observe that the above cone $C_{\bar u}^\tau$ is not equal to the cone $E_{\bar u}^\tau$ defined in Theorem 2.4. However, if $v \in C_{\bar u}^\tau$, then

$$|J'(\bar u)v| = \int_\Gamma |\bar d(x)v(x)|\,dx \leq \tau \int_{\{x:|\bar d(x)|\leq\tau\}} |v(x)|\,dx \leq \tau\sqrt{|\Gamma|}\|v\|_{L^2(\Gamma)}.$$

Thus, we have that $C_{\bar u}^\tau \subset E_{\bar u}^{\tau_\Gamma}$, with $\tau_\Gamma = \tau\sqrt{|\Gamma|}$. Hence, Theorem 2.4 can be applied.

4 A Bang-Bang Control Problem

The reader is referred to [4] for proofs and extensions of the results stated below. Let Ω be an open and bounded domain in $\mathbb{R}^n$, $n \leq 3$, with a Lipschitz boundary Γ. In this domain, we consider the following control problem

$$(\mathrm{P}_2) \quad \begin{cases} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x))\,dx \\ \alpha \leq u(x) \leq \beta \end{cases}$$

where y_u is the solution of the Dirichlet problem

$$\begin{cases} -\Delta y + f(y) = u \text{ in } \Omega, \\ \qquad\qquad y = 0 \text{ on } \Gamma, \end{cases} \tag{4.1}$$

$-\infty < \alpha < \beta < +\infty$ and L and f satisfy the following assumptions.
Assumption (D1) The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is of class C^2 and $f'(t) \geq 0$ for every $t \in \mathbb{R}$.
Assumption (D2) The function $L : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable with respect to the first variable and of class C^2 with respect to the second. Moreover, $L(\cdot, 0) \in L^1(\Omega)$, and for all $M > 0$ there is a constant $C_{L,M} > 0$ and a function $\psi_M \in L^{\bar p}(\Omega)$ such that

$$\left|\frac{\partial L}{\partial y}(x,y)\right| \leq \psi_M(x), \quad \left|\frac{\partial^2 L}{\partial y^2}(x,y)\right| \leq C_{L,M}.$$

For every $M > 0$ and $\varepsilon > 0$ there exists $\delta > 0$, depending on M and ε such that

$$\left|\frac{\partial^2 L}{\partial y^2}(x,y_2) - \frac{\partial^2 L}{\partial y^2}(x,y_1)\right| < \varepsilon \text{ if } |y_1|, |y_2| \leq M,\ |y_2 - y_1| \leq \delta, \text{ for a.a. } x \in \Omega.$$

Hereafter, we will denote

$$\mathcal{K} = \{u \in L^\infty(\Omega) : \alpha \le u(x) \le \beta \ \text{ for a.e. } \ x \in \Omega\}.$$

For every $u \in L^p(\Omega)$, with $p > n/2$, the state equation (4.1) has a unique solution $y_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$. The proof of this result is a quite standard combination of Schauder's fixed point theorem and the $L^\infty(\Omega)$ estimates [12]. For the continuity of the solution in $\bar{\Omega}$ see, for instance, [10, Theorem 8.30]. Moreover, the mapping $G : L^p(\Omega) \longrightarrow H_0^1(\Omega) \cap C(\bar{\Omega})$, with $G(u) = y_u$, is of class C^2. In the sequel, we will take $p = 2$ and we will denote by $z_v = G'(u)v$ the solution of

$$\begin{cases} -\Delta z + f'(y_u)z = v \text{ in } \Omega, \\ \qquad\qquad\quad z = 0 \text{ on } \Gamma, \end{cases} \tag{4.2}$$

where $y_u = G(u)$ is the state corresponding to u. As usual, we consider the adjoint state equation associated with a control u

$$\begin{cases} -\Delta \varphi + f'(y_u)\varphi = \dfrac{\partial L}{\partial y}(x, y_u) \text{ in } \Omega, \\ \qquad\qquad\quad \varphi = \quad 0 \qquad \text{on } \Gamma, \end{cases} \tag{4.3}$$

denoted by φ_u. Because of the assumptions on L, we have that $\varphi \in H_0^1(\Omega) \cap C(\bar{\Omega})$. Moreover, there exists $M > 0$ such that

$$\|y_u\|_\infty + \|\varphi_u\|_\infty \le M \quad \forall u \in \mathcal{K}. \tag{4.4}$$

Under the above assumptions, the problem (P$_2$) has at least one solution $\bar{u}$ with an associated state $\bar{y} \in H_0^1(\Omega) \cap C(\bar{\Omega})$. The cost functional $J : L^2(\Omega) \longrightarrow \mathbb{R}$ is of class C^2 and the first and second derivatives are given by

$$J'(u)v = \int_\Omega \varphi_u(x)v(x)\,dx, \tag{4.5}$$

and

$$J''(u)(v_1, v_2) = \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x, y_u(x)) - \varphi_u(x)f''(y_u(x))\right) z_{v_1}(x)z_{v_2}(x)\,dx, \tag{4.6}$$

where $z_{v_i} = G'(v_i)$ are the solution of (4.2) for $v = v_i$, $i = 1, 2$.

Any local solution $\bar{u}$ satisfies the optimality system

$$\begin{cases} -\Delta \bar{y} + f(\bar{y}) = \bar{u} \text{ in } \Omega, \\ \qquad\qquad\quad \bar{y} = 0 \text{ on } \Gamma, \end{cases} \tag{4.7}$$

$$\begin{cases} -\Delta \bar{\varphi} + f'(\bar{y})\bar{\varphi} = \dfrac{\partial L}{\partial y}(x, \bar{y}) \text{ in } \Omega, \\ \qquad\qquad\quad \bar{\varphi} = \quad 0 \qquad \text{on } \Gamma, \end{cases} \tag{4.8}$$

$$\int_\Omega \bar{\varphi}(x)(u(x) - \bar{u}(x))\,dx \ge 0 \quad \forall u \in \mathcal{K}. \tag{4.9}$$

From the last condition, we deduce as usual for a.a. $x \in \Omega$

$$\bar{u}(x) \begin{cases} = \alpha \text{ if } \bar{\varphi}(x) > 0, \\ = \beta \text{ if } \bar{\varphi}(x) < 0, \end{cases} \quad \text{and} \quad \bar{\varphi}(x) \begin{cases} > 0 \text{ if } \bar{u}(x) = \alpha, \\ < 0 \text{ if } \bar{u}(x) = \beta, \\ = 0 \text{ if } \alpha < \bar{u}(x) < \beta. \end{cases} \tag{4.10}$$

The cone of critical directions associated with $\bar{u}$ is defined by

$$C_{\bar{u}} = \{v \in L^2(\Omega) : v(x) \begin{cases} \geq 0 \text{ if } \bar{u}(x) = \alpha \\ \leq 0 \text{ if } \bar{u}(x) = \beta \\ = 0 \text{ if } \bar{\varphi}(x) \neq 0 \end{cases} \}$$

Then, the necessary second order condition satisfied is written in the form

$$J''(\bar{u})v^2 \geq 0 \quad \forall v \in C_{\bar{u}}. \tag{4.11}$$

For the above results the reader is referred to [5] or [6], where similar cases were studied. Let us remark that in the case where the set of zeros of $\bar{\varphi}$ has a zero Lebesgue measure, then $\bar{u}(x)$ is either α or β for almost all points $x \in \Omega$, i.e. $\bar{u}$ is a bang-bang control. Moreover, in this case, $C_{\bar{u}} = \{0\}$, therefore (4.11) does not provide any information. Consequently, it is unlikely that the sufficient second order conditions could be based on the set $C_{\bar{u}}$. To overcome this drawback we are going to increase the set $C_{\bar{u}}$. For every $\tau \geq 0$ we define

$$C^\tau_{\bar{u}} = \{v \in L^2(\Omega) : v(x) \begin{cases} \geq 0 \text{ if } \bar{u}(x) = \alpha \\ \leq 0 \text{ if } \bar{u}(x) = \beta \\ = 0 \text{ if } |\bar{\varphi}(x)| > \tau \end{cases} \}$$

It is obvious that $C^0_{\bar{u}} = C_{\bar{u}}$. An example due to Dunn [9] proves that, in general, the second order condition based on the cone $C_{\bar{u}}$ is not sufficient for the local optimality. Before analyzing (P_2), let us take a look on its Tikhonov regularization. For any $\Lambda > 0$, let us consider the problem

$$(\mathrm{P}_{2,\Lambda}) \qquad \min_{u \in \mathcal{K}} J_\Lambda(u) = \int_\Omega L(x, y_u(x))\,dx + \frac{\Lambda}{2}\int_\Omega u^2(x)\,dx.$$

Then, we have

$$J'_\Lambda(u)v = \int_\Omega (\varphi_u + \Lambda u)v\,dx$$

and

$$J''_\Lambda(u)(v_1, v_2) = \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x, y_u) - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u)\right) z_{v_1} z_{v_2}\,dx + \Lambda \int_\Omega v_1 v_2\,dx.$$

Now, we apply Theorem 2.2 to ($\mathrm{P}_{2,\Lambda}$) and we obtain the following result.

Theorem 4.1. *Let $\bar{u} \in \mathcal{K}$ satisfy that*

$$J'_\Lambda(\bar{u})(u - \bar{u}) \geq 0 \quad \forall u \in \mathcal{K} \quad \textit{and}$$
$$J''_\Lambda(\bar{u})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\}.$$

Then, there exists $\delta > 0$ and $\varepsilon > 0$ such that

$$J_\Lambda(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \le J_\Lambda(u) \quad \forall u \in B_2(\bar{u};\varepsilon) \cap \mathcal{K}.$$

In the above theorem and hereafter, $B_2(\bar{u};\varepsilon)$ denotes the $L^2(\Omega)$-ball of center at $\bar{u}$ and radius ε. Now, invoking Theorem 2.4 and observing that $C^\tau_{\bar{u}} \subset E^{\tau_\Omega}_{\bar{u}}$ for $\tau_\Omega = \sqrt{|\Omega|}\tau$, we get the following theorem.

Theorem 4.2. *Let $\bar{u} \in \mathcal{K}$ satisfy $J'_\Lambda(\bar{u})(u - \bar{u}) \ge 0$ for every $u \in \mathcal{K}$. Then, the following assumptions are equivalent*

1. $J''_\Lambda(\bar{u})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\}$.
2. $\exists \nu > 0$ *and* $\tau > 0$ *s.t.* $J''_\Lambda(\bar{u})v^2 \ge \nu\|v\|^2_{L^2(\Omega)} \quad \forall v \in C^\tau_{\bar{u}}$.
3. $\exists \nu > 0$ *and* $\tau > 0$ *s.t.* $J''_\Lambda(\bar{u})v^2 \ge \nu\|z_v\|^2_{L^2(\Omega)} \quad \forall v \in C^\tau_{\bar{u}}$,

where $z_v = G'(\bar{u})v$.

In the case $\Lambda = 0$, Dunn's example shows that *1* is not enough, in general, to assure the local optimality of $\bar{u}$. We will see below that *2* does not hold for $\Lambda = 0$. Then, it remains to analyze if the assumption *3* is enough for the local optimality of $\bar{u}$ when $\Lambda = 0$. The next theorem proves that it is sufficient.

Theorem 4.3. *Let us assume that $\bar{u}$ is a feasible control for problem* (P$_2$) *satisfying the first order optimality conditions* (4.7)-(4.9) *and suppose that there exist $\delta > 0$ and $\tau > 0$ such that*

$$J''(\bar{u})v^2 \ge \delta\|z_v\|^2_{L^2(\Omega)} \quad \forall v \in C^\tau_{\bar{u}}, \tag{4.12}$$

where $z_v = G'(\bar{u})v$ is the solution of (4.2) *for $y = \bar{y}$. Then, there exists $\varepsilon > 0$ such that*

$$J(\bar{u}) + \frac{\delta}{8}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)} \le J(u) \quad \forall u \in B_2(\bar{u};\varepsilon) \cap \mathcal{K}, \tag{4.13}$$

with $z_{u-\bar{u}} = G'(\bar{u})(u - \bar{u})$.

Corollary 4.1 *Under the hypotheses of Theorem 4.3, there exists $\varepsilon > 0$ such that*

$$J(\bar{u}) + \frac{\delta}{9}\|y_u - \bar{y}\|^2_{L^2(\Omega)} \le J(u) \quad \forall u \in B_2(\bar{u};\varepsilon) \cap \mathcal{K}. \tag{4.14}$$

We finish by showing that the statement *2* of Theorem 4.2 does not hold for $\Lambda = 0$. Indeed, let us assume that it holds. Then, a simple modification of the proof of Theorem 4.3, see [4], leads to the inequality

$$J(\bar{u}) + \frac{\nu}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \le J(u) \quad \forall u \in B_2(\bar{u};\varepsilon) \cap \mathcal{K}, \tag{4.15}$$

for some $\nu > 0$ and $\varepsilon > 0$. Then, $\bar{u}$ is a solution of the problem

$$(\mathrm{P}_\nu) \qquad \min_{u \in B_2(\bar{u};\varepsilon) \cap \mathcal{K}} J(u) - \frac{\nu}{2}\int_\Omega (u - \bar{u})^2\, dx.$$

The Hamiltonian of this control problem is given by

$$H(x, y, u, \varphi) = L(x, y) + \varphi(u - f(x, y)) - \frac{\nu}{2}(u - \bar{u}(x))^2.$$

From the Pontryagin's principle we deduce

$$H(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)) = \min_{t \in [\alpha, \beta]} H(x, \bar{y}(x), t, \bar{\varphi}(x)) \text{ for almost all } x \in \Omega.$$

However, invoking (4.10) we obtain that this is a contradiction to the following facts that can be easily checked

$$\begin{cases} \text{If } 0 < \bar{\varphi}(x) < \dfrac{\nu}{2}(\beta - \alpha) \text{ then } H(x, \bar{y}(x), \beta, \bar{\varphi}(x)) < H(x, \bar{y}(x), \alpha, \bar{\varphi}(x)), \\ \text{If } 0 > \bar{\varphi}(x) > \dfrac{\nu}{2}(\alpha - \beta) \text{ then } H(x, \bar{y}(x), \alpha, \bar{\varphi}(x)) < H(x, \bar{y}(x), \beta, \bar{\varphi}(x)). \end{cases}$$

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