On existence, uniqueness, and convergence, of optimal control problems governed by parabolic variational inequalities

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Abstract. I) We consider a system governed by a free boundary problem with Tresca condition on a part of the boundary of a material domain with a source term g through a parabolic variational inequality of the second kind. We prove the existence and uniqueness results to a family of distributed optimal control problems over g for each parameter $h > 0$, associated to the Newton law (Robin boundary condition), and of another distributed optimal control problem associated to a Dirichlet boundary condition. We generalize for parabolic variational inequalities of the second kind the Mignot's inequality obtained for elliptic variational inequalities (Mignot, J. Funct. Anal., 22 (1976), 130-185), and we obtain the strictly convexity of a quadratic cost functional through the regularization method for the non-differentiable term in the parabolic variational inequality for each parameter h. We also prove, when $h \to +\infty$, the strong convergence of the optimal controls and states associated to this family of optimal control problems with the Newton law to that of the optimal control problem associated to a Dirichlet boundary condition.
II) Moreover, if we consider a parabolic obstacle problem as a system governed by a parabolic variational inequalities of the first kind then we can also obtain the same results of Part I for the existence, uniqueness and convergence for the corresponding distributed optimal control problems.
III) If we consider, in the problem given in Part I, a flux on a part of the boundary of a material domain as a control variable (Neumann boundary optimal control problem) for a system governed by a parabolic variational inequality of second kind then we can also obtain the existence and uniqueness results for Neumann boundary optimal control problems for each parameter $h > 0$, but in this case the convergence when $h \to +\infty$ is still an open problem.

Keywords: Parabolic variational inequalities, convex combination of solutions, regularization method, optimal control problems, strict convexity of cost functional.

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1 Introduction

The goal of this paper is to show the existence and uniqueness results to a family of distributed (see Sections 2 and 3) or Neumann boundary (see Section 4) optimal control problems for each parameter $h > 0$, associated to the Newton law (Robin boundary condition on a part of the boundary of the material domain), and of another distributed optimal control problem associated to a Dirichlet boundary condition. The system of these optimal control problems are governed by free boundary problems (with Tresca boundary condition (see Sections 2 and 4) or of an obstacle type problem (see Section 3) through a parabolic variational inequalities of the first (see Section 3) or second (see Sections 2 and 4) kind [2], [6]. An optimal control problem for elliptic variational inequality of the second kind is given in [9].

In order to prove the existence and uniqueness results we generalize for parabolic variational inequalities of the second kind the Mignot's inequality obtained for elliptic variational inequalities [18], and then we obtain the strictly convexity of a quadratic cost functional through the regularization method for the non-differentiable term for each parameter $h > 0$.

We also prove, when $h \to +\infty$, the strong convergence of the optimal controls and states associated to this family of optimal control problems with the Newton law to that of the optimal control problem associated to a Dirichlet boundary condition.

We obtain these convergence without using the adjoint state which is a great advantage with respect to the proof given previously for optimal control problems governed by elliptic and parabolic variational equalities [3], [11], [12], [17].

These convergence when $h \to +\infty$ are valid for the optimal control problems given in Sections 2 and 3, and it is still an open problem for the Neumann boundary optimal control problem given in Section 4.

2 Distributed optimal control problems governed by parabolic variational inequality of second kind

Let Ω a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset$, and $meas(\Gamma_1) > 0$. We set $V = H^1(\Omega)$, $V_0 = \{v \in V : v_{|\Gamma_1} = 0\}$, $H = L^2(\Omega)$, $\mathcal{H} = L^2(0,T;H)$, $\mathcal{V} = L^2(0,T;V)$, and the closed convex set $K_b = \{v \in V : \quad v_{|\Gamma_1} = b\}$. Let given

$$\begin{aligned} b \in L^2(0,T;H^{1/2}(\Gamma_1)), \quad b > 0, \quad g \in \mathcal{H}, \quad g \geq 0, \\ q \in L^2((0,T) \times \Gamma_2), \quad q > 0, \quad u_b \in K_b. \end{aligned} \tag{1}$$

We consider the following variational problems [6]

Problem 1. Let given g, q, b and u_b as in (1). Find $u = u_g \in \mathcal{C}(0,T,H) \cap L^2(0,T;K_b)$ with $\dot{u} \in \mathcal{H}$, such that $u(0) = u_b$, and solution of the parabolic variational inequality of second kind:

$$< \dot{u}, v - u > + a(u, v-u) + \Phi(v) - \Phi(u) \geq (g, v-u), \quad \forall v \in K_b, \quad t \in (0,T).$$

Problem 2. Let given g, q, b and u_b as in (1). For all $h > 0$, find $u = u_{hg}$ in $\mathcal{C}(0,T,H) \cap \mathcal{V}$ with $\dot{u} \in \mathcal{H}$, such that $u(0) = u_b$, and solution of the parabolic variational inequality of second kind

$$< \dot{u}, v-u > + a_h(u, v-u) + \Phi(v) - \Phi(u) \geq (g, v-u) \\ +h \int_{\Gamma_1} b(v-u)ds, \; \forall v \in V, \quad t \in (0,T).$$

Where $\dot{u} = u_t$, $<,>$ denotes the duality brackets between V' and V, a is a symmetric, continuous and coercive bilinear form over V_0, and Φ is given by

$$\Phi(v) = \int_{\Gamma_2} q|v|ds, \tag{2}$$

and

$$a(u,v) = \int_\Omega \nabla u \nabla v dx, \quad a_h(u,v) = a(u,v) + h\int_{\Gamma_1} uvds, \quad (g,v) = \int_\Omega gvdx.$$

Moreover from [15], [20], [21] we have that:

$$\exists \lambda_1 > 0 \quad \text{such that } \lambda_h \|v\|_V^2 \leq a_h(v,v) \quad \forall v \in V, \text{ with } \lambda_h = \lambda_1 \min\{1\,,\,h\}$$

that is, a_h is also a bilinear continuous, symmetric and coercive form on V.

We remark that on $\Gamma_1 \times (0,T)$, *Problem* 1 is with the Dirichlet condition $u_{|_{\Gamma_1}} = b$, while *Problem* 2 is with the Robin's condition $-\nabla u \cdot n = h(u-b)$, where n is the exterior unit vector normal to Γ. The functional Φ comes from the Tresca condition on Γ_2 [1], [4].
The existence and uniqueness of the solution to each of the above *Problem* 1 and *Problem* 2, is well known see for example [7], [8], [10]. Therefore, it allows us to consider $g \mapsto u_g$ as a function from $\mathcal{H}$ to $\mathcal{C}(0,T,H) \cap \mathcal{V}$.

Let $M > 0$ be a constant and $\mathcal{H}_+ = \{g \in \mathcal{H} : g \geq 0\}$. We consider the following distributed optimal control problems defined by:

$$\text{Find } g_{op} \in \mathcal{H}_+ \quad \text{such that} \quad J(g_{op}) = \min_{g \in \mathcal{H}_+} J(g), \tag{3}$$

$$\text{Find } g_{op_h} \in \mathcal{H}_+ \quad \text{such that} \quad J(g_{op_h}) = \min_{g \in \mathcal{H}_+} J_h(g), \tag{4}$$

where the cost functional $J : \mathcal{H} \to \mathbb{R}$ and $J_h : \mathcal{H} \to \mathbb{R}$ such that [16] (see also [13], [14], [22])

$$J(g) = \frac{1}{2}\|u_g\|_{\mathcal{H}}^2 + \frac{M}{2}\|g\|_{\mathcal{H}}^2, \quad \text{and} \quad J_h(g) = \frac{1}{2}\|u_{hg}\|_{\mathcal{H}}^2 + \frac{M}{2}\|g\|_{\mathcal{H}}^2, \tag{5}$$

being here u_g, u_{hg} the unique solutions of the parabolic variational *Problem* 1, and *Problem* 2 respectively, and corresponding to the control g in $\mathcal{H}$. In order to prove the strict convexity of the cost functional J and J_h, we generalize for parabolic variational inequalities a main property [18] that : For any two control $g_i \in \mathcal{H}$, $i = 1$ or $i = 2$, we have

$$u_{\mu g_1+(1-\mu)g_2} \le \mu u_{g_1} + (1-\mu)u_{g_2}, \quad \forall \mu \in [0,1],$$

$$u_{h(\mu g_1+(1-\mu)g_2)} \le \mu u_{hg_1} + (1-\mu)u_{hg_2}, \quad \forall \mu \in [0,1],$$

by using a regularization method for the non-differentiable functional Φ (see [6]). Then we prove the following

Theorem 1. *[6] Let $u_{hg_{op_h}}$, g_{op_h} and $u_{g_{op}}$, g_{op} be the states and the optimal controls defined in Problem 1 and Problem 2 respectively. Then, we obtain the following asymptotic behavior:*

$$\lim_{h\to+\infty} \|u_{hg_{op_h}} - u_{g_{op}}\|_{\mathcal{V}} = 0, \tag{6}$$

$$\lim_{h\to+\infty} \|g_{op_h} - g_{op}\|_{\mathcal{H}} = 0. \tag{7}$$

3 Distributed optimal control problems governed by parabolic variational inequality of first kind

We will examine in this section, some distributed optimal control problems, for which the strong formulation can be linked to a free boundary problems of complementarity type (Obstacle problems [19]), given for example by the following conditions:

$$u \ge 0, \quad u(\dot{u} - \Delta u - g) = 0, \quad \dot{u} - \Delta u - g \ge 0 \quad \text{in } \Omega, \tag{8}$$

$$u = b \ge 0 \text{ on } \Gamma_1, \quad -\frac{\partial u}{\partial n} = f \text{ on } \Gamma_2, \quad \text{and} \quad u(0) = u_b \tag{9}$$

and

$$u \ge 0, \quad u(\dot{u} - \Delta u - g) = 0, \quad \dot{u} - \Delta u - g \ge 0 \quad \text{in } \Omega, \tag{10}$$

$$-\frac{\partial u}{\partial n} = h(u-b) \text{ on } \Gamma_1, \quad -\frac{\partial u}{\partial n} = f \text{ on } \Gamma_2, \quad \text{and} \quad u(0) = u_b \tag{11}$$

where Ω is a multidimentional regular domain whose boundary is $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let consider the convex set K_b as in Section 2. It is classical that, for a given positive $b \in L^2(0,T;H^{\frac{1}{2}}(\Gamma_1))$, $f \in L^2(0,T;L^2(\Gamma_2))$, and $g \in \mathcal{H}$, the variational formulations of *Problems* (8)-(9) and (10)-(11) are respectively given by the following parabolic variational problems:

Problem 3. Let given g, b and u_b as in (1) and $f \in L^2(0,T;L^2(\Gamma_2))$. Find $u = u_g \in \mathcal{C}(0,T,H) \cap L^2(0,T;K_b)$ with $\dot{u} \in \mathcal{H}$, such that $u(0) = u_b$, and

$$< \dot{u}, v-u > + a(u, u-v) \ge (g, v-u) - \int_{\Gamma_2} f(v-u)ds, \quad \forall v \in K_b, \forall t \in (0,T).$$

Problem 4. Find $u = u_{hg} \in \mathcal{C}(0, T, H) \cap \mathcal{V}$ with $\dot{u} \in \mathcal{H}$, such that $u(0) = u_b$, and

$$< \dot{u}, v - u > +a_h(u, u - v) \geq (g, v - u) + h \int_{\Gamma_1} b(v - u)ds$$
$$- \int_{\Gamma_2} f(v - u)ds, \quad \forall v \in V, \quad \forall t \in (0, T).$$

where a and a_h are as in Section 2. Then the existence and uniqueness of the solution to *Problem* 3 and *Problem* 4, is also well known see for example [7], [8], [10]. Then it allows us to consider $g \mapsto u_g$ as a function from $\mathcal{H}$ to $\mathcal{C}(0, T, H) \cap \mathcal{V}$. Let $M > 0$ be a constant. We consider the same family of distributed optimal control problems (3)-(4) and we obtain the same results of the previous Theorem 1.

Theorem 2. *Let g, b, u_b as in (1) and $f \leq 0$ in $\Gamma_2 \times (0, T)$, we can obtain the same results as in Section 2, for the corresponding distributed optimal control problems (3)-(4) when $g \geq 0$ is the control variable.*

4 Neumann boundary optimal control problem governed by parabolic variational inequalities of second kind

We assume in this section that the boundary of a multidimensional regular domain Ω is decomposed in three parts $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with $meas(\Gamma_1) > 0$ and $meas(\Gamma_3) > 0$.

We consider a Neumann boundary optimal control problem whose system is governed by a free boundary problem with Tresca conditions on a portion Γ_2 of the boundary, with the flux f on Γ_3 *as the control variable*, given by:

Problem 5.

$$\dot{u} - \Delta u = g \quad \text{in} \quad \Omega \times (0, T),$$

$$\left| \frac{\partial u}{\partial n} \right| < q \Rightarrow u = 0, \text{ on } \Gamma_2 \times (0, T),$$
$$\left| \frac{\partial u}{\partial n} \right| = q \Rightarrow \exists k > 0 : \quad u = -k \frac{\partial u}{\partial n}, \text{ on } \Gamma_2 \times (0, T),$$

$$u = b \quad \text{on} \quad \Gamma_1 \times (0, T),$$

$$-\frac{\partial u}{\partial n} = f \quad \text{on} \quad \Gamma_3 \times (0, T),$$

with the initial condition

$$u(0) = u_b \quad \text{on} \quad \Omega,$$

and the compatibility condition on $\Gamma_1 \times (0, T)$

$$u_b = b \quad \text{on} \quad \Gamma_1 \times (0, T),$$

where $q > 0$ is the Tresca friction coefficient on Γ_2 ([1], [4], [10]). We define the space $\mathcal{F} = L^2(0, T; L^2(\Gamma_3))$.

The variational formulation of *Problem* 5 leads to the following parabolic variational problem:

Problem 6. Let given g, q, b and u_b as in (1) and $f \in \mathcal{F}$, $f \leq 0$. Find $u = u_f$ in $\mathcal{C}(0, T, H) \cap L^2(0, T; K_b)$ with $\dot{u} \in \mathcal{H}$, such that $u(0) = u_b$, and for $t \in (0, T)$

$$< \dot{u}, v - u > +a(u, u - v) + \Phi(v) - \Phi(u) \geq (g, v - u) - \int_{\Gamma_3} f(v - u)ds, \ \forall v \in K_b.$$

where a and Φ are defined as in Section 2.

We consider also the following problem where we change, in *Problem* 5, only the Dirichlet condition on $\Gamma_1 \times (0, T)$ by the Newton law or a Robin boundary condition.

Problem 7.

$$\dot{u} - \Delta u = g \quad \text{in} \quad \Omega \times (0, T),$$

$$\left|\frac{\partial u}{\partial n}\right| < q \Rightarrow u = 0, \ \text{on } \Gamma_2 \times (0, T),$$
$$\left|\frac{\partial u}{\partial n}\right| = q \Rightarrow \exists k > 0 : \quad u = -k\frac{\partial u}{\partial n}, \ \text{on } \Gamma_2 \times (0, T),$$

$$-\frac{\partial u}{\partial n} = h(u - b) \quad \text{on} \quad \Gamma_1 \times (0, T),$$

$$-\frac{\partial u}{\partial n} = f \quad \text{on} \quad \Gamma_3 \times (0, T),$$

with the initial condition

$$u(0) = u_b \quad \text{on} \quad \Omega,$$

and the condition of compatibility on $\Gamma_1 \times (0, T)$

$$u_b = b \quad \text{on} \quad \Gamma_1 \times (0, T).$$

The variational formulation of the problem (7) leads to the the following parabolic variational problem

Problem 8. Let given g, q, b, u_b and f as in *Problem* 6. For all $h > 0$, find $u = u_{hf} \in \mathcal{C}(0, T, H) \cap \mathcal{V}$ with $\dot{u} \in \mathcal{H}$, such that $u(0) = u_b$, and for $t \in (0, T)$

$$\begin{aligned} < \dot{u}, v - u > +a_h(u, u - v) + \Phi(v) - \Phi(u) \geq (g, v - u) - \int_{\Gamma_3} f(v - u)ds \\ +h\int_{\Gamma_1} b(v - u)ds, \quad \forall v \in V, \end{aligned}$$

where a_h and Φ are defined as in Section 2.

4.1 Neumann boundary optimal control problems

Let $M > 0$ be a constant and we define the space $\mathcal{F}_- = \{f \in \mathcal{F} : f \leq 0\}$. We consider the new following Neumannn boundary optimal control problems defined by:

Problem 9. Find the optimal control $f_{op} \in \mathcal{F}_-$ such that

$$J(f_{op}) = \min_{f \in \mathcal{F}_-} J(f) \tag{12}$$

where the cost functional $J : \mathcal{F} \to \mathbb{R}_0^+$ is given by

$$J(f) = \frac{1}{2}\|u_f\|_{\mathcal{H}}^2 + \frac{M}{2}\|f\|_{\mathcal{F}}^2 \quad (M > 0) \tag{13}$$

and u_f is the unique solution of the *Problem* 6.

Problem 10. Find the optimal control $f_{op_h} \in \mathcal{F}_-$ such that

$$J(f_{op_h}) = \min_{f \in \mathcal{F}_-} J_h(f) \tag{14}$$

where the cost functional $J_h : \mathcal{F} \to \mathbb{R}_0^+$ is given by

$$J_h(f) = \frac{1}{2}\|u_{hf}\|_{\mathcal{H}}^2 + \frac{M}{2}\|f\|_{\mathcal{F}}^2 \quad (M > 0, \quad h > 0) \tag{15}$$

and u_{hf} is the unique solution of *Problem* 8.

Theorem 3. *Under the assumptions given in Problem 6, we have the following properties:*

a) *The cost functional J is strictly convex on $\mathcal{F}_-$,*
b) *There exists a unique optimal $f_{op} \in \mathcal{F}_-$ solution of the new Neumannn boundary optimal control Problem 9.*

Proof. We give some sketch of the proof.

i) We generalize for parabolic variational inequalities of the second kind the estimates obtained for convex combination of solutions for elliptic variational inequalities [5] that is, the estimate between

$$u_4(\mu) = u_{\mu f_1 + (1-\mu) f_2}, \text{ and } u_3(\mu) = \mu u_{f_1} + (1-\mu) u_{f_2},$$

for any two element f_1 and f_2 in $\mathcal{F}$.

ii) The main difficulty, to prove this result comes from the fact that the functional Φ is not differentiable. To overcome this difficulty, we use the regularization method and consider for $\varepsilon > 0$ the following approach of Φ defined by:

$$\Phi_\varepsilon(v) = \int_{\Gamma_2} q\sqrt{\varepsilon^2 + |v|^2} ds, \qquad \forall v \in V, \tag{16}$$

which is Gateaux differentiable, with

$$\langle \Phi'_\varepsilon(w)\,,\, v\rangle = \int_{\Gamma_2} \frac{qwv}{\sqrt{\varepsilon^2+|w|^2}} ds \qquad \forall (w,v)\in V^2.$$

We define u^ε as the unique solution of the corresponding parabolic variational inequality for all $\varepsilon > 0$. We obtain that for all $\mu \in [0,1]$ we have $u_4^\varepsilon(\mu) \leq u_3^\varepsilon(\mu)$ for all $\varepsilon > 0$.

iii) When $\varepsilon \to 0$ we have that:

$$u_i^\varepsilon \to u_i \text{ strongly in } \mathcal{V}\cap L^\infty(0,T;H) \text{ for } i=1,2,3,4, \tag{17}$$

for all $\mu \in [0,1]$ and therefore we get:

$$0 \leq u_4(\mu) \leq u_3(\mu) \quad in \quad \Omega\times[0,T], \quad \forall \mu\in[0,1]. \tag{18}$$

iv) For all $\mu \in]0,1[$, and for all f_1, f_2 in $\mathcal{F}$, and by using $f_3(\mu) = \mu f_1 + (1-\mu) f_2$ we obtain that:

$$\mu J(f_1) + (1-\mu)J(f_2) - J(f_3(\mu)) = \frac{1}{2}\left(\|u_3(\mu)\|^2_{\mathcal{H}} - \|u_4(\mu)\|^2_{\mathcal{H}}\right)$$
$$+\frac{1}{2}\mu(1-\mu)\|u_{f_1}-u_{f_2}\|^2_{\mathcal{H}} + \frac{M}{2}\mu(1-\mu)\|f_1-f_2\|^2_{\mathcal{F}}. \tag{19}$$

Then J is strictly convex functional on $\mathcal{F}_-$ and therefore there exists a unique optimal $f_{op} \in \mathcal{F}_-$ solution of the new Neumannn boundary optimal control Problem 9.

Theorem 4. *Under the assumptions given in Problem 6, we have the following properties:*

a) *The cost functional J_h are strictly convex on $\mathcal{F}_-$, for all $h > 0$,*
b) *There exists a unique optimal $f_{op_h} \in \mathcal{F}_-$ solution of the new Neumannn boundary optimal control Problem 10, for all $h > 0$.*

Proof. We follow a similar method to the one developed in Theorem 3 for all $h > 0$.

4.2 Open problem

The convergence of the new Neumann boundary optimal control *Problem* 10 to the new Neumann boundary optimal control *Problem* 9 when $h \to \infty$ is an open problem.

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