Dynamic Contact Problem for Viscoelastic von Kármán-Donnell Shells

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Abstract. We deal with initial-boundary value problems describing vertical vibrations of viscoelastic von Kármán-Donnell shells with a rigid inner obstacle. The short memory (Kelvin-Voigt) material is considered. A weak formulation of the problem is in the form of the hyperbolic variational inequality. We solve the problem using the penalization method.

Keywords: Von Kármán-Donnell shell, unilateral dynamic contact, viscoelasticity, solvability, penalty approximation

1 Introduction

Contact problems represent an important but complex topic of applied mathematics. Its complexity profounds if the dynamic character of the problem is respected. For elastic problems there is only a very limited amount of results available (cf. [3] and there cited literature). Viscosity makes possible to prove the existence of solutions for a broader set of problems for membranes, bodies as well as for linear models of plates. The presented results extend the research made in [2], where the problem for a viscoelastic short memory von Kármán plate in a dynamic contact with a rigid obstacle was considered. Our results also extend the research made for the quasistatic contact problems for viscoelastic shells (cf. [1]). A thin isotropic shallow shell occupies the domain

$$G = \{(x,z) \in R^3 : \ x = (x_1,x_2) \in \Omega, \ |z - \mathcal{Z}| < h/2\},$$

where $h > 0$ is the thickness of the shell, $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain in $\mathbb{R}$ with a sufficiently smooth boundary Γ. We set $I \equiv (0,T)$ a bounded time interval, $Q = I \times \Omega, \ S = I \times \Gamma$. The unit outer normal vector is denoted by $\mathbf{n} = (n_1, n_2), \ \tau = (-n_2, n_1)$ is the unit tangent vector. The displacement is denoted by $\mathbf{u} \equiv (u_i)$. The strain tensor is defined as

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_3 \partial_j u_3) - k_{ij} u_3 - x_3 \partial_{ij} u_3, \ i,j = 1,2$$

with $k_{12} = k_{21} = 0$ and the curvatures $k_{ii} > 0,\ i = 1, 2$.

Further, we set

$$[u, v] \equiv \partial_{11}u\partial_{22}v + \partial_{22}u\partial_{11}v - 2\partial_{12}u\partial_{12}v.$$

In the sequel, we denote by $W_p^k(M)$, $k \geq 0$, $p \in [1, \infty]$ the Sobolev spaces defined on a domain or an appropriate manifold M. By $\mathring{W}_p^k(M)$ the spaces with zero traces are denoted. If $p = 2$ we use the notation $H^k(M)$, $\mathring{H}^k(M)$. The duals to $\mathring{H}^k(M)$ are denoted by $H^{-k}(M)$. For the anisotropic spaces $W_p^k(M)$, $k = (k_1, k_2) \in \mathbb{R}_+^2$, k_1 is related with the time variable while k_2 with the space variables. We shall use also the Bochner-type spaces $W_p^k(I; X)$ for a time interval I and a Banach space X. Let us remark that for $k \in (0, 1)$ their norm is defined by the relation

$$\|w\|^p_{W_p^k(I;X)} \equiv \int_I \|w(t)\|^p_X dt + \int_I \int_I \frac{\|w(t) - w(s)\|^p_X}{|s - t|^{1+kp}} ds\, dt.$$

By $C(M)$ we denote the spaces of continuous functions on a (possibly relatively) compact manifold M. They are equipped with the max-norm. Analogously the spaces $C(M; X)$, are introduced for a Banach space X. The following generalization of the Aubin's compactness lemma verified in [4] Theorem 3.1 will be essentially used:

Lemma 1 *Let* $B_0 \hookrightarrow\hookrightarrow B \hookrightarrow B_1$ *be Banach spaces, the first reflexive and separable. Let* $1 < p < \infty,\ 1 \leq r < \infty$. *Then*

$$W \equiv \{v;\ v \in L_p(I; B_0),\ \dot{v} \in L_r(I, B_1)\} \hookrightarrow\hookrightarrow L_p(I; B).$$

2 Short memory material

2.1 Problem formulation

Employing the Einstein summation, the constitutional law has the form

$$\sigma_{ij}(\mathbf{u}) = \frac{E_1}{1 - \mu^2}\partial_t\big((1-\mu)\varepsilon_{ij}(\mathbf{u}) + \mu\delta_{ij}\varepsilon_{kk}(\mathbf{u})\big) + \frac{E_0}{1 - \mu^2}\big((1-\mu)\varepsilon_{ij}(\mathbf{u}) + \mu\delta_{ij}\varepsilon_{kk}(\mathbf{u})\big).$$

The constants $E_0,\ E_1 > 0$ are the Young modulus of elasticity and the modulus of viscosity, respectively. We shall use the abbreviation $b = h^2/(12\varrho(1 - \mu^2))$, where $h > 0$ is the shell thickness and ϱ is the density of the material. We involve the rotation inertia expressed by the term $a\Delta\ddot{u}$ in the first equation of the considered system with $a = \frac{h^2}{12}$. It will play the crucial role in the deriving a strong convergence of the sequence of velocities $\{\dot{u}_m\}$ in the appropriate space. We assume the shell clamped on the boundary. We generalize the dynamic elastic model due to the von Kármán-Donnell theory mentioned in [6]. The classical formulation for the deflection $u_3 \equiv u$ and the Airy stress function v is then the initial-value problem

$$\left.\begin{array}{l}\ddot{u} + a\Delta\ddot{u} + b(E_1\Delta^2\dot{u} + E_0\Delta^2 u) - [u,v] - \Delta_k * v = f + g,\\ u - \Psi \geq 0,\ g \geq 0,\ (u-\Psi)g = 0,\\ \Delta^2 v + E_1\partial_t(\frac{1}{2}[u,u] + k_{11}\partial_{22}u + k_{22}\partial_{11}u)\\ +E_0(\frac{1}{2}[u,u] + \Delta_k u) = 0\end{array}\right\} \text{ on } Q, \tag{1}$$

$$u = \partial_n u = v = \partial_n v = 0 \text{ on } S, \tag{2}$$

$$u(0,\cdot) = u_0,\ \dot{u}(0,\cdot) = u_1 \text{ on } \Omega. \tag{3}$$

The obstacle function $\Psi \in L_\infty(\Omega)$ is fulfilling $0 < U_0 \leq u_0 - \Psi$ in Ω and

$$\Delta_k u \equiv \partial_{11}(k_{22}u) + \partial_{22}(k_{11}u), \tag{4}$$

$$\Delta_k^* v \quad \equiv k_{22}\partial_{11}v + k_{11}\partial_{22}v. \tag{5}$$

We define the operators $L : H^2(\Omega) \to \mathring{H}^2(\Omega),\ \Phi : H^2(\Omega)\times H^2(\Omega) \to \mathring{H}^2(\Omega)$ by uniquely solved equations

$$(\Delta Lu, \Delta w) \equiv (\Delta_k u, w)\,\forall w \in \mathring{H}^2(\Omega), \tag{6}$$

$$(\Delta\Phi(u,v), \Delta w) \equiv ([u,v], w)\,\forall w \in \mathring{H}^2(\Omega). \tag{7}$$

with the inner product $(\cdot,\cdot)$ in the space $L_2(\Omega)$. The operator L is linear and compact. The bilinear operator Φ is symmetric and compact. Moreover due to Lemma 1 from [5] $\Phi : H^2(\Omega)^2 \to W_p^2(\Omega),\ 2 < p < \infty$ and

$$\|\Phi(u,v)\|_{W_p^2(\Omega)} \leq c\|u\|_{H^2(\Omega)}\|v\|_{W_p^1(\Omega)}\ \forall u \in H^2(\Omega),\, v \in W_p^1(\Omega). \tag{8}$$

We have also $L : H^2(\Omega) \mapsto W_p^2(\Omega),\ \ 2 < p < \infty$ and

$$\|Lu\|_{W_p^2(\Omega)} \leq c\|u\|_{H^2(\Omega)}\ \forall u \in H^2(\Omega). \tag{9}$$

For $u, y \in L_2(I; H^2(\Omega))$ we define the bilinear form A by

$$A(u,y) := b\big(\partial_{kk}u\partial_{kk}y + \mu(\partial_{11}u\partial_{22}y + \partial_{22}u\partial_{11}y) + 2(1-\mu)\partial_{12}u\partial_{12}y\big).$$

We introduce shifted cone $\mathcal{K}$ by

$$\mathcal{K} := \{y \in H^{1,2}(Q);\ \dot{y} \in L_2(I, \mathring{H}^1(\Omega);\ y \geq \Psi\}. \tag{10}$$

Then the variational formulation of the problem (1-3) has the form of

Problem $\mathcal{P}$. *Find* $u \in \mathcal{K}$ *such that* $\dot{u} \in L_2(I; \mathring{H}^2(\Omega))$ *and*

$$\begin{array}{l}\int_Q (E_1 A(\dot{u}, y-u) + E_0 A(u, y-u))\,dx\,dt\\ +\int_Q [u, E_1\partial_t(\frac{1}{2}\Phi(u,u) + Lu) + E_0(\frac{1}{2}\Phi(u,u) + Lu)](y-u)\,dx\,dt\\ +\int_Q \Delta_k\left(E_1\partial_t(\frac{1}{2}\Phi(u,u) + Lu) + E_0(\frac{1}{2}\Phi(u,u)+Lu)\right)(y-u)\,dx\,dt\\ -\int_Q (a\nabla\dot{u}\cdot\nabla(\dot{y}-\dot{u}) + \dot{u}(\dot{y}-\dot{u}))\,dx\,dt\\ +\int_\Omega (a\nabla\dot{u}\cdot\nabla(y-u) + \dot{u}(y-u))(T,\cdot)\,dx\\ \geq \int_\Omega (a\nabla u_1\cdot\nabla(y(0,\cdot)-u_0) + u_1(y(0,\cdot)-u_0))\,dx\\ +\int_Q f(y_1-u)\,dx\,dt\ \forall y \in \mathcal{K}.\end{array} \tag{11}$$

2.2 The penalization

For any $\eta > 0$ we define the *penalized problem*
Problem $\mathcal{P}_\eta$. *Find* $u \in H^{1,2}(Q)$ *such that* $\dot u \in L_2(I; \mathring{H}^2(\Omega))$, $\ddot u \in L_2(I; \mathring{H}^1(\Omega))$,

$$\begin{aligned}&\int_Q \big(\ddot u z + a\nabla \ddot u \cdot \nabla z + E_1 A(\dot u, z) + E_0 A(u,z)\big)\,dx\,dt\\ &+ \int_Q [u, E_1\partial_t(\tfrac12 \Phi(u,u) + Lu) + E_0(\tfrac12\Phi(u,u)+Lu)]z\,dx\,dt\\ &+\int_Q \Delta_k\,\big(E_1\partial_t(\tfrac12\Phi(u,u)+Lu)+E_0(\tfrac12\Phi(u,u)+Lu)\big)\,z\,dx\,dt\\ &=\int_Q (f+\eta^{-1}(u-\Psi)^-)z\,dx\,dt\ \forall z\in L_2(I;H^2(\Omega))\end{aligned} \tag{12}$$

and the conditions (3) remain valid.

Lemma 2 *Let* $f \in L_2(Q)$, $u_0 \in \mathring{H}^2(\Omega)$, *and* $u_1 \in \mathring{H}^1(\Omega)$. *Then there exists a solution* u *of the problem* $\mathcal{P}_\eta$.

Proof. Let us denote by $\{w_i \in \mathring{H}^2(\Omega);\ i = 1,2,...\}$ a basis of $\mathring{H}^2(\Omega)$ orthonormal in $H^1(\Omega)$ with respect to the inner product

$$(u,v)_a = \int_\Omega (uv + a\nabla u\cdot\nabla v)\,dx,\ \ u,v\in H^1(\Omega).$$

We construct the Galerkin approximation u_m of a solution in a form

$$u_m(t) = \sum_{i=1}^m \alpha_i(t) w_i,\ \ \alpha_i(t)\in\mathbb{R},\ \ i=1,...,m,\ \ m\in N, \tag{13}$$

$$\begin{aligned}&(\ddot u_m(t), w_i)_a + \int_\Omega \big(E_1 A(\dot u_m(t), w_i) + E_0 A(u_m(t), w_i)\big)\,dx +\\ &\int_\Omega \Delta\big(E_1\partial_t(\tfrac12\Phi(u_m,u_m)+Lu_m) + E_0(\tfrac12\Phi(u_m,u_m)+Lu_m)\big)\\ &\times\Delta(\Phi(u_m,w_i)+Lw_i)\,dx\\ &=\int_\Omega \big(f(t)+\eta^{-1}(u_m(t)-\Psi)^-\big)w_i\,dx,\ \ i=1,...,m,\end{aligned} \tag{14}$$

$$u_m(0)=u_{0m},\ \ \dot u_m(0)=u_{1m},\ \ u_{0m}\to u_0 \text{ in } \mathring{H}^2(\Omega),\ \ u_{1m}\to u_1 \text{ in } \mathring{H}^1(\Omega). \tag{15}$$

After multiplying the equation (14) by $\dot\alpha_i(t)$, summing up with respect to i , taking in mind the definitions of the operators Φ, L and integrating we obtain the *a priori* estimates not depending on m:

$$\begin{aligned}&\|\dot u_m\|^2_{L_2(I;\mathring{H}^2(\Omega))} + \|\dot u_m\|^2_{L_\infty(I;\mathring{H}^1(\Omega))} + \|u_m\|^2_{L_\infty(I;\mathring{H}^2(\Omega))}\\ &+\|\partial_t\Phi(u_m,u_m)\|^2_{L_2(I;\mathring{H}^2(\Omega))} + \|\partial_t L u_m\|^2_{L_2(I;\mathring{H}^2(\Omega))}\\ &+\eta^{-1}\|(u_m-\Psi)^-\|_{L_\infty(I;L_2(\Omega))} \le c \equiv c(f,u_0,u_1).\end{aligned} \tag{16}$$

Moreover the estimates (8), (9) imply

$$\|\partial_t\Phi(u_m,u_m)\|_{L_2(I;W^2_p(\Omega))} + \|\partial_t L u_m\|_{L_2(I;W^2_p(\Omega))} \le c_p\ \forall p>2. \tag{17}$$

After multiplying the equation (14) by $\ddot\alpha_i(t)$, summing up and integrating we obtain the estimate of $\ddot u_m$

$$\|\ddot u_m\|_{L_2(I;H^1(\Omega))} \le c_\eta,\ \ m\in\mathbb{N}. \tag{18}$$

Applying the estimates (16)-(18), the compact imbedding theorem and the interpolation, we obtain for any $p \in [1,\infty)$, a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$), a function u and the convergences

$$\begin{array}{l}
\ddot{u}_m \rightharpoonup \ddot{u} \text{ in } L_2(I;H^1(\Omega)),\\
\dot{u}_m \rightharpoonup^* \dot{u} \text{ in } L_\infty(I;\mathring{H}^1(\Omega)),\\
\dot{u}_m \rightharpoonup \dot{u} \text{ in } L_2(I;\mathring{H}^2(\Omega)),\\
\dot{u}_m \to \dot{u} \text{ in } L_p(I;\mathring{H}^(\Omega)) \cap L_\infty(I;H^{2-\varepsilon}(\Omega)) \ \forall \varepsilon > 0,\\
u_m \to u \text{ in } C(\bar{I};W^1_p(\Omega),\\
\partial_t\big(\tfrac{1}{2}\Phi(u_m,u_m)+Lu_m\big) \to \partial_t\big(\tfrac{1}{2}\Phi(u,u)+Lu\big) \text{ in } L_2(I;W^2_p(\Omega))
\end{array} \tag{19}$$

implying that a function u fulfils the identity (12). The initial conditions (3) follow due to (15) and the proof of the existence of a solution is complete.

2.3 Solving the original problem

We verify the existence theorem

Theorem 1 *Let $f \in L_2(Q)$, $u_i \in \mathring{H}^2(\Omega)$, $i = 0,\ 1,\ 0 < U_0 \le u_0 - \Psi$. Then there exists a solution of the Problem $\mathcal{P}$.*

Proof. We perform the limit process for $\eta \to 0$. We write u_η for the solution of the problem $\mathcal{P}_{1,\eta}$. The *a priori* estimates (16) imply the estimates

$$\begin{array}{l}
\|\dot{u}_\eta\|^2_{L_2(I;\mathring{H}^2(\Omega))} + \|\dot{u}_\eta\|^2_{L_\infty(I;\mathring{H}^1(\Omega))} + \|u_\eta\|^2_{L_\infty(I;\mathring{H}^2(\Omega))}\\
+\|\partial_t\Phi(u_\eta,u_\eta)\|^2_{L_2(I;W^2_p(\Omega))} + \|\partial_t L u_\eta\|^2_{L_2(I;W^2_p(\Omega))}\\
+\eta^{-1}\|(u_\eta-\Psi)^-\|_{L_\infty(I;L_2(\Omega))} \le c_p,\ p>2.
\end{array} \tag{20}$$

To get the crucial estimate for the penalty, we put $z = u_0 - u_\eta(t,\cdot)$ in (12) and obtain the estimate

$$\begin{array}{c}
0 \le U_0 \int_Q \eta^{-1}(u_\eta-\Psi)^- dx\,dt \le \int_Q \|\eta^{-1}(u_\eta-\Psi)^-(u_0-\Psi)dx\,dt\\
\le \int_Q \|\eta^{-1}(u_\eta-\Psi)^-(u_0-u_\eta)dx\,dt\\
= \int_Q \big(\dot{u}_\eta^2 + a|\nabla\dot{u}_\eta|^2 + A(\,(E_1\partial_t u_\eta + E_0 u_\eta), u_0-u_\eta)\\
+E_1\partial_t(\Delta(Lu_\eta+\tfrac{1}{2}\Phi(u_\eta,u_\eta)))\Delta(L(u_0-u_\eta)+\Phi(u_\eta,u_0-u_\eta))\\
+E_0\Delta(Lu_\eta+\tfrac{1}{2}\Phi(u_\eta,u_\eta))\Delta(L(u_0-u_\eta)+\Phi(u_\eta,u_0-u_\eta))\big)\,dx\,dt\\
-\int_Q f(u_0-u_\eta)\,dx\,dt + \int_\Omega \big(\big(\dot{u}_\eta(u_0-u_\eta)+a\nabla\dot{u}_\eta\cdot\nabla(u_0-u_\eta)\big)(T,\cdot)\big)\ dx.
\end{array}$$

Applying the *a priori* estimates (20) we obtain

$$\|\eta^{-1}u_\eta^-\|_{L_1(Q)} \le c(f,u_0,u_1,\Psi). \tag{21}$$

With respect to Dirichlet conditions we obtain from (12) and (21) the dual estimate

$$\|-a\Delta\ddot{u}_\eta + \ddot{u}_\eta\|_{L_1(I;H^{-2}(\Omega))} \le c. \tag{22}$$

We take the sequence $\{u_k\} \equiv \{u_{\eta_k}\},\ \eta_k \to 0+$.

After applying the Lemma 1 with the spaces

$$B_0 = L_2(\Omega),\ B = H^{-1}(\Omega),\ B_1 = H^{-2}(\Omega)$$

we obtain the relative compactness of the sequence $\{-a\Delta\dot{u}_k + \dot{u}_k\}$ in $L_2(I; H^{-1}(\Omega)$ and with the help of the test function $\dot{u}_k - \dot{u}$ the crucial strong convergence

$$\dot{u}_k \to \dot{u} \text{ in } L_2(I; \mathring{H}^1(\Omega)). \tag{23}$$

Simultaneously we have the convergences

$$\begin{aligned} &\dot{u}_k \rightharpoonup \dot{u} \text{ in } L_2(I; \mathring{H}^2(\Omega)),\\ &\dot{u}_k \to \dot{u} \text{ in } L_2(I; W_p^1(\Omega)),\\ &\tfrac{1}{2}\partial_t\Phi(u_k, u_k) + \partial_t L u_k \rightharpoonup \tfrac{1}{2}\partial_t\Phi(u, u) + \partial_t L u \text{ in } L_2(I; W_p^2(\Omega)). \end{aligned} \tag{24}$$

It can be verified after inserting the test function $z = y - u_k$ in (12) for $y \in \mathcal{K}$, performing the integration by parts in the terms containing $\ddot{u}$, applying the convergences (23), (24), using the definitions of the operators $L,\ \Phi$ in (6), (7) and the weak lower semicontinuity that the limit function u is a solution of the original problem $\mathcal{P}$.

Remark 1. The existence Theorem 1 can be after some modification verified also for another types of boundary conditions.

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