On an Algorithm for Dynamic Reconstruction in Systems with Delay in Control

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Abstract. We discuss a problem of the dynamic reconstruction of unknown input controls in nonlinear vector equations. A regularizing algorithm is proposed for reconstructing these controls simultaneously with the processes. The algorithm is stable with respect to informational noises and computational errors.

Keywords: dynamic reconstruction, method of auxiliary models

1 Introduction. Problem statement.

Consider a controlled system described by the following equation

$$\dot{x}(t) = f_1(t, u_t(s), x_t(s)) + f_2(t, x_t(s))u(t) \tag{1}$$

with the initial state

$$u_{t_0}(s) = u_0(s) \in C([-\tau^u_m, 0]; R^{n_1}), \qquad x_{t_0}(s) = x_0(s) \in C([-\tau^x_n, 0]; R^{n_2}). \tag{2}$$

Here t is time from a fixed interval $T = [t_0, \vartheta]$ ($t_0 < \vartheta < +\infty$); $x(t) = (x_1(t), \ldots, x_{n_2}(t))$ is the phase state of the system; $u(t) = (u_1(t), \ldots, u_{n_1}(t))$ is a control; the symbols $x_t(s)$ and $u_t(s)$ mean the functions $x_t(s) = x(t+s)$ for $s \in [-\tau^x_n, 0]$ and $u_t(s) = u(t+s)$ for $s \in [-\tau^u_m, 0]$, respectively. We assume that initial state (2) is Lipschitz. For simplicity, we assume also that the initial state $x_0(s)$, $u_0(s)$ is fixed and known. The control $u = u(t) = (u_1(t), \ldots, u_{n_1}(t))$ is called an admissible control if its components $u_i(t)$, $i \in [1 : n_1]$, are Lebesgue measurable functions on the interval T and values $u(t)$ belong to a given compact set P from Euclidean space R^{n_1} for almost all $t \in T$. The set of all admissible controls is denoted by $P(\cdot)$. Therefore, $P(\cdot) = \{u(\cdot) \in L_2(T; R^{n_1}) : u(t) \in P$ for a. a. $t \in T\}$. By the trajectory (or the solution) $x(\cdot)$ of equation (1) with

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initial state (2) corresponding to some admissible control $u(\cdot)$, we call absolutely continuous on T function $x = x(t)$ satisfying (1) for a.a. $t \in T$.

Condition 1. The elements of matrix function

$$f_{2ij}(t, x_t(s)) = f_{2ij}(t, x(t), x(t-\tau_1^x), \ldots, x(t-\tau_n^x)), \quad i \in [1:n_2], \quad j \in [1:n_1],$$

and vector-valued function

$$f_{1i}(t, u_t(s), x_t(s)) =$$
$$= f_{1i}(t, u(t-\tau_1^u), \ldots, u(t-\tau_m^u), x(t), x(t-\tau_1^x), \ldots, x(t-\tau_n^x)), \quad i \in [1:n_2]$$

satisfy the Lipschitz conditions

$$|f_{2ij}(t_1, x_0^{(1)}, x_1^{(1)}, \ldots, x_n^{(1)}) - f_{2ij}(t_2, x_0^{(1)}, x_1^{(2)}, \ldots, x_n^{(2)})| \leq \tag{3}$$
$$\leq C_1(|t_2 - t_1| + \sum_{j=0}^{n} |x_j^{(1)} - x_j^{(2)}|),$$

$$|f_{1i}(t_1, u_1^{(1)}, \ldots, u_m^{(1)}, x_0^{(1)}, x_1^{(1)}, \ldots, x_n^{(1)}) - f_{1i}(t_2, u_1^{(2)}, \ldots, u_m^{(2)}, x_0^{(2)}, x_1^{(2)}, \ldots, x_n^{(2)})|$$
$$\leq d_1(|t_2 - t_1| + \sum_{i=1}^{m} |u_i^{(1)} - u_i^{(2)}| + \sum_{j=0}^{n} |x_j^{(1)} - x_j^{(2)}|). \tag{4}$$

In this case, under this condition for any pair, i.e., for initial state (2) and the control $u(\cdot) \in P(\cdot)$, there exists a unique solution of equation (1).

Let $u(\cdot)$ be an admissible control realizing during the given time interval T; $x(\cdot)$ be the real motion generated by this control. We assume that the phase states $x(\tau_i)$ of the system are inaccurately measured at frequent enough time moments $\tau_i \in T$ in the process. Measurement results $\xi^h(\tau_i) \in R^{n_2}$ satisfy the inequalities

$$|\xi^h(\tau_i) - x(\tau_i)| \leq h. \tag{5}$$

Here, the quantity $h \in (0,1)$ specifies the measurement error.

In the present paper, we construct an algorithm that reconstructs the control $u(\cdot)$ on the basis of the current information $\xi^h(\cdot)$ in real time. Since the exact reconstruction is impossible due to the error of measurements $\xi^h(\cdot)$ we require that the algorithm should generate some approximation. Namely, it is required to construct an algorithm allowing us, on the basis of the inaccurate measurements $\xi^h(\cdot)$, and in real time, to form the admissible control $v^h(\cdot)$ such that the mean-square deviation of $v^h(\cdot)$ from $u(\cdot)$:, i.e.,

$$|v^h(\cdot) - u(\cdot)|^2_{L_2(T)} = \int_{t_0}^{\vartheta} |v^h(t) - u(t)|^2 \, dt, \tag{6}$$

is arbitrarily small for the sufficiently small measurement error h. Since the measurements are inaccurate it is in general impossible to identify $u(t)$ precisely, therefore the problem is to approximate the input by some function $v^h(t)$.

Here and below, the symbol $|\cdot|$ stands for both the Euclidean norm and the corresponding matrix norm and for the modulo of a number. In what follows, we set $\tau_m^u = \tau_n^x = \tau$ for simplicity, and by $\xi^h(\cdot)$ we denote the function $\xi^h(t)$, $t \in [t_0 - \tau, \vartheta]$ such that $\xi^h(t) = x_0(t - t_0)$ for $t \in [t_0 - \tau, t_0)$, $\xi^h(t) = \xi^h(\tau_i)$ for $t \in [\tau_i, \tau_{i+1})$, $i \in [0 : d-1]$, where $\tau_i = \tau_{h,i}$, $d = d_h$, $\xi^h(\tau_i)$ satisfies (5).

The suggested solution outline is the following ([1–6]). An auxiliary control system (model M) described by equation of the form

$$\dot{w}(t) = F(t, \xi_t^h(s), v_t^h(s)), \quad w_{t_0}(s) = w_0(s), \quad t \in T \tag{7}$$

is associated with the real dynamical system (1). Here the vector $w \in R^{n_2}$ characterizes state of the model, the form of function F is corrected below, vector v^h is control action. After that, the problem of reconstruction of input $u(\cdot)$ is replaced by the problem of positional control of the model. This process is realized on the time interval T in such a way that control $v^h(\cdot)$ "approximates" appropriately $u(\cdot)$. First, one takes a uniform net $\Delta = \{\tau_i\}_{i=0}^m$, $\tau_{i+1} = \tau_i + \delta$, $\delta > 0$, $i \in [0 : m]$, $\tau_0 = 0$, $\tau_m = T$ with the step δ. Then, on the interval $t \in [\tau_i, \tau_{i+1})$ the model is acted upon the controls

$$v_i^h = V_h(\tau_i, w_{\tau_i}(s), \xi_{\tau_i}^h(s)) \tag{8}$$

calculated at the moment τ_i by use of some rule, which hereinafter we shall identify with mapping V_h. Thus, the controls in the model are realized by the method of feedback control. Its value on the interval $[\tau_i, \tau_{i+1}]$ depends on the measurement results $\xi^h(\cdot)$ corresponding to the phase state $x(\cdot)$ of the system (1) and state w of the model (7). The described process forms the piece-wise function

$$v^h(t) = v_i^h, \qquad t \in [\tau_i, \tau_{i+1})$$

in real time synchro with the motion of real system (1). Thus, to solve the problem above, we should specify a model and a control law for this model.

2 Algorithm for solving the problem

As a model, we take the following system of linear ordinary differential equation

$$\dot{w}(t) = f_1(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s)) + f_2(\tau_i, \xi_{\tau_i}^h(s))v_i^h + 2(\xi^h(\tau_i) - w(\tau_i)), \tag{9}$$

$$w \in R^{n_2}, \quad t \in [\tau_i, \tau_{i+1}), \quad \tau_i = \tau_{h,i}, \quad v_{t_0}^h(s) = u_0(s),$$

with the initial state $w(t_0) = \xi^h(t_0)$. The solution of this equation $w(\cdot) = w(\cdot; t_0, w_{t_0}(s), v^h(\cdot))$ is understood in the sense of Caratheodory. So, the right-hand side of equation of the model (7) has the form

$$F(t, \xi_t^h(s), v_t^h(s)) = f_1(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s)) + f_2(\tau_i, \xi_{\tau_i}^h(s))v_i^h +$$

$$+2(\xi^h(\tau_i) - w(\tau_i)), \quad t \in [\tau_i, \tau_{i+1}).$$

Introduce the following notation: $\varDelta^{(j)} = [t_j, t_{j+1}]$, $\quad t_j = t_0 + \tau_1^x j$; the symbol l stands for the integer part of the number τ/τ_1^x; $j_* = \max\{j : t_j < \vartheta\}$,

$$g_j(h) = h^{(1/3)^j}, \quad j \in [1 : j_*].$$

Fix a partition of the interval T with a step $\delta = \delta(h)$ depending on the measurement error h, i.e.,

$$\varDelta_h = \{\tau_{h,i}\}_{i=0}^{d_h}, \quad \tau_i = \tau_{h,i}, \quad \tau_{h,0} = t_0, \quad \tau_{h,d_h} = \vartheta, \tag{10}$$

(for simplicity, we assume that $\tau_i - \tau_{i-1} = \delta = \delta(h)$). Without loss of generality, we can suppose that the partition $\varDelta_h$ is chosen in such a way that $t_j \in \varDelta_h$. Define the law of forming the control v_i^h in the model (for $\tau_i \in [t_j, t_{j+1}) \cap T$) by the relations

$$V_h(\tau_i, w_{\tau_i}(s), \xi^h_{\tau_i}(s)) = V_j(\tau_i, w_{\tau_i}(s), \xi^h_{\tau_i}(s))$$

$$= \arg\min\{2(l_i, f_2(\tau_i, \xi^h_{\tau_i}(s))v) + \alpha_j |v|^2 : v \in P\}. \tag{11}$$

Here α_j is a parameter, $j \in [0 : j_*]$, $l_i = w(\tau_i) - \xi^h(\tau_i)$.

Condition 2. Let $n_2 \geq n_1$, and let there exists a number $c_* > 0$ such that the matrix $f_2(t, x_t(s))$ has a minor of order n_1 with the property: the $n_1 \times n_1$-dimensional matrix $\bar{f}_2(t) = \bar{f}_2(t, x_t(s))$ corresponding to this minor satisfies the inequality

$$|\bar{f}_2(t)u| \geq c_* |u|$$

for each $t \in T$ and all $u \in R^{n_1}$.

We choose the parameter α_j which plays the role of the regularizer, as follows:

$$\alpha_0 = Ch^{2/3}, \quad \alpha_j = C g_j^{2/3}(h), \quad j \geq 1, \quad C = \text{const} > 0. \tag{12}$$

Let us describe the algorithm for solving the problem above.

Before the initial moment the value h and the partition $\varDelta = \varDelta_h$ with diameter $\delta = \delta(h)$ are fixed. The work of the algorithm starting at time $t = 0$ is decomposed into $m_h - 1$ steps. At the i-th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following actions take place. First, at time moment τ_i vector v_i^h is calculated by formula (11). Then the control $v^h(t) = v_i^h$ is fed onto the input of the model (9). After that, we transform the state $w_{\tau_i}(s)$ of the model into $w_{\tau_{i+1}}(s)$. The procedure stops at time ϑ.

The following theorem is true.

Theorem 1. *Let $\delta = \delta(h) \leq h$. Then the inequalities*

$$\nu^{(j)} \equiv |v^h(\cdot) - u(\cdot)|^2_{L_2(\varDelta^{(j-1)}; R^{n_1})} \leq c_j g_j(h), \quad j \in [1 : j_*],$$

are valid. Here, $v^h(t) = u(t)$ for $t \in [t_0 - \tau, t_0]$, $v^h(t) = u_0(-\tau)$ for $t \in [t_0 - \tau - \tau_1^u, t_0 - \tau)$.

The proof of the theorem is based on auxiliary statements, which are used in forthcoming considerations. Introduce two systems

$$\dot p(t) = f_1(t) + f_2(t)u_1(t), \quad t \in T,$$
$$\dot q(t) = F_1(t) + F_2(t)u_2(t),$$

where $p(t)$, $q(t) \in R^n$, $f_1(\cdot)$, $F_1(\cdot) \in L_2(T; R^n)$, $f_2(\cdot) \in L_2(T; R^{n\times r})$, $F_2(\cdot) \in L_2(T; R^{n\times r})$, $u_1(\cdot)$, $u_2(\cdot) \in L_2(T; R^r)$, $|u_l(\cdot)|_{L_\infty(T;R^r)} \le K$, $l = 1, 2$.

Introduce the notation: $\Delta_*^{(j)} = [t_j^*, t_{j+1}^*] \cap T$, $t_j^* = t_0 + \tau_* j$, $j \in [0 : j_0]$, $\Delta^{(-1)} = [t_0 - \tau_*, t_0]$, $\tau_* = \mathrm{const} \in (0, \vartheta - t_0)$, $j_0 = \max\{j : t_j^* \le \vartheta\}$. Let $r \le n$ and let there exists a number $c > 0$ such that the matrix $f_2(t)$ has a minor of order r such that the $r \times r$-matrix $\bar f_2(t)$ corresponding to this minor satisfies the following inequality: $|\bar f_2(t)u| \ge c|u|$ for each $t \in T$ and all $u \in R^r$.

It is easy to verify the following lemmas.

Lemma 1. *Let the function $t \to (\bar f_2(t))^{-1}u_1(t)$ be a function of bounded variation on T and let the conditions*

$$|f_1(\cdot) - F_1(\cdot)|^2_{L_2(\Delta_*^{(j)};R^n)} \le a_1^{(j)}, \quad |f_2(\cdot) - F_2(\cdot)|^2_{L_2(\Delta_*^{(j)};R^{n\times r})} \le a_2^{(j)},$$

$$|p(t) - q(t)|^2 + \tilde\alpha_j \int\limits_{t_j^*}^{t} \{|u_2(\nu)|^2 - |u_1(\nu)|^2\}\, d\nu \le a_3^{(j)} \qquad t \in [t_j^*, t_{j+1}^*],$$

$$|p(t_j^*) - q(t_j^*)|^2 \le a_4^{(j)}, \quad \tilde\alpha_j = \mathrm{const} \in (0, +\infty)$$

be true. Then the inequality

$$|u_1(\cdot) - u_2(\cdot)|^2_{L_2(\Delta_*^{(j)};R^r)} \le K_j\{\sum_{l=1}^{4} (a_l^{(j)})^{1/2} + \tilde\alpha_j^{1/2}\} + a_3^{(j)}/\tilde\alpha_j$$

is valid.

Lemma 2. *The bunches of solutions of systems* (1) *and* (9) *are bounded in the space $W^{1,\infty}(T; R^{n_2}) = \{x(\cdot) \in L_2(T; R^{n_2}); \dot x(\cdot) \in L_2(T; R^{n_2})\}$.*

We use the relation

$$\varepsilon_j(t) = |x(t) - w(t)|^2 + \alpha_j \int\limits_{t_j}^{t} \{|v^h(\nu)|^2 - |u(\nu)|^2\}\, d\nu, \quad j \in [0 : j_*], \quad t \in T.$$

Lemma 3. *The following inequalities*

$$\varepsilon_j(t) \le b_j, \quad t \in \Delta^{(j)} \cap T, \quad j \in [0 : j_*],$$

are valid, where

$$b_j = |x(t_j) - w(t_j)|^2 + c_j^{(1)}(h + \delta) + c_j^{(2)} \sum_{k=j-l}^{j} \nu^{(k)},$$

$c_j^{(1)}$, $c_j^{(2)}$ are some constants, which can be explicitly written.

Proof. Fix $\tau_i \in \Delta^{(j)}$. Then for $t \in \Delta^{(j)} \cap \delta_i = [\tau_i, \tau_{i+1}]$, we obtain

$$\varepsilon_j(t) \le \varepsilon_j(\tau_i) + \sum_{j=1}^{4} \Lambda_{ji}(t), \tag{13}$$

where

$$\Lambda_{1i}(t) = 2(s_i, \int_{\tau_i}^{t} \{f_1(\nu, u_\nu(s), x_\nu(s)) - f_1(\tau_i, v^h_\nu(s), \xi^h_{\tau_i}(s))\}\, d\nu),\ \ s_i = x(\tau_i) - w(\tau_i),$$

$$\Lambda_{2i}(t) = 2(s_i, \int_{\tau_i}^{t} \{f_2(\nu, x_\nu(s))u(\nu) -$$

$$- f_2(\tau_i, \xi^h_{\tau_i}(s))v^h_i\}\, d\nu) + \alpha_j \int_{\tau_i}^{t} \{|v^h(\nu)|^2 - |u(\nu)|^2\}\, d\tau,$$

$$\Lambda_{3i}(t) = -2(t-\tau_i)(s_i, \xi^h(\tau_i) - w(\tau_i)), \quad \Lambda_{4i}(t) = (t-\tau_i) \int_{\tau_i}^{t} |\dot{w}(\tau) - \dot{x}(\tau)|^2\, d\tau.$$

By virtue of lemma 2, we have

$$\Lambda_{4i}(t) \le K^{(j)}_*(t-\tau_i)^2, \quad t \in \delta_i. \tag{14}$$

Note that $v^h(\tau_i + s) = v^h(t+s)$ for $s \ge t_0 - \tau_i$, $t \in [\tau_i, \tau_{i+1}]$ and in addition

$$|\xi^h(\tau_i + s) - x(t+s)| \le K_*(h + t - \tau_i) \quad \text{for} \quad \tau_i + s \ge t_0 - \tau. \tag{15}$$

Taking into account lemma 2, as well as the Lipschitz property of the functions $u_0(s)$ and $x_0(s)$, inequalities (4) and the relation

$$|\xi^h(\tau_i + s) - x(t+s)| \le K_*(h + t - \tau_i) \quad \text{for} \quad \tau_i + s \ge t_0 - \tau, \tag{16}$$

we obtain for $t \in \delta_i$ the estimate

$$\int_{\tau_i}^{t} |f_1(\nu, u_\nu(s), x_\nu(s)) - f_1(\tau_i, v^h_\nu(s), \xi^h_{\tau_i}(s))|\, d\nu \le$$

$$\le K^{(j)}_1(t-\tau_i)(h+t-\tau_i) + K^{(j)}_2(t-\tau_i)^{1/2} \sum_{k=1}^{m} \Big(\int_{\tau_i - \tau^u_k}^{t - \tau^u_k} |u(\nu) - v^h(\nu)|^2\, d\nu \Big)^{1/2}.$$

Here, $\tau^x_0 = 0$. In this case, the inequality

$$\Lambda_{1i}(t) \le 2(t-\tau_i)|x(\tau_i) - w(\tau_i)|^2 + K^{(j)}_3\{(t-\tau_i)(h+t-\tau_i)^2 +$$

$$+\sum_{k=1}^{m}\int\limits_{\tau_i-\tau_k^u}^{t-\tau_k^u}|u(\nu)-v^h(\nu)|^2\,d\nu\} \tag{17}$$

holds for $t\in\delta_i$. In view of (5), we have

$$\Lambda_{3i}(t)\le -2(t-\tau_i)|x(\tau_i)-w(\tau_i)|^2+K_4^{(j)}h(t-\tau_i),\quad t\in\delta_i. \tag{18}$$

Moreover, from (5), (3), and (16), we derive

$$|f_2(\nu,x_\nu(s))u(\nu)-f_2(\tau_i,\xi^h_{\tau_i}(s))u(\nu)|\le K_0(h+\nu-\tau_i)$$

for $\nu\in[\tau_i,\tau_{i+1}]$. In this case,

$$\Lambda_{2i}(t)\le K_5^{(j)}(t-\tau_i)(h+t-\tau_i)+$$

$$+\int\limits_{\tau_i}^{t}\{2(l_i,f_2(\tau_i,\xi^h_{\tau_i}(s))\{v_i^h-u(\nu)\}+\alpha_j\{|v_i^h|^2-|u(\nu)|^2\}\}\,d\nu.$$

The rule for forming the control v_i^h (11) and the last inequality imply

$$\Lambda_{2i}(t)\le K_5^{(j)}(t-\tau_i)(h+t-\tau_i). \tag{19}$$

Finally, taking into account (13)–(19), we conclude that for $t\in\Delta^{(j)}\cap\delta_i$

$$\varepsilon_j(t)\le\varepsilon_j(\tau_i)+K_6^{(j)}\delta(h+\delta)+K_3^{(j)}\sum_{k=1}^{m}\int\limits_{\tau_i-\tau_k^u}^{t-\tau_k^u}|u(\nu)-v^h(\nu)|^2\,d\nu,$$

i.e., for $t\in\Delta^{(j)}=[t_j,t_{j+1}]$,

$$\varepsilon_j(t)\le\varepsilon_j(t_j)+K_7^{(j)}(h+\delta)+K_8^{(j)}\int\limits_{t_j-\tau}^{t_{j+1}-\tau_1^u}|u(\nu)-v^h(\nu)|^2\,d\nu.$$

Note that $\tau=l\tau_1^u+\gamma$, $\gamma\ge 0$. Therefore, $t_{j+1}-\tau_1^u=t_j$, $t_{j-l-1}\le t_j-\tau\le t_{j-l}$. In this case, for $t\in\Delta^{(j)}$ we have

$$\varepsilon_j(t)\le\varepsilon_j(t_j)+K_7^{(j)}(h+\delta)+K_9^{(j)}\sum_{k=j-l}^{j}\nu^{(k)}.$$

Here, constants $K_k^{(j)}$, $k\in[0:9]$ are written explicitly. Thus, one can assume that $c_j^{(1)}=K_7^{(j)}$ and $c_j^{(2)}=K_9^{(j)}$. The lemma is proved.

Lemma 4. *Let $\delta\le h$ and values α_j be given by* (12). *Then the inequalities*

$$\nu^{(j)}\le c_j g_j(h), \tag{20}$$

$$b_j\le c_j^{(0)}g_j(h) \tag{21}$$

are valid.

Proof. For simplicity, set $t_{j_*+1} = \vartheta$. By virtue of lemma 3, we have for $t \in \Delta^{(j)}$

$$|x(t)-w(t)| \le \Big(\varepsilon_j(t)+\alpha_j \int_{t_j}^{t} \{|v^h(\nu)|^2+|u(\nu)|^2\}\, d\nu\Big)^{1/2} \le \Big(b_j+\alpha_j\rho_A\Big)^{1/2}, \quad (22)$$

where $\rho_A = 2\tau_* d^2(P)$ and $d(P) = \sup\{|u| : u \in P\}$. Taking into account the inclusion $t_j \in \Delta_h$, we conclude that for any $j \in [0 : j_*]$, one can specify the number $i = i_j(h)$ such that $t_j = \tau_{i_j(h)}$. Introduce the notation $\varrho_j \equiv |f_1(\cdot) - F_1(\cdot)|^2_{L_2(\Delta^{(j)};R^{n_2})}$. In this case, by virtue of lemma 2, as well as of (4) and (16), we obtain

$$\varrho_j \le d_j^{(1)} \sum_{i=i_j(h)}^{i=i_{j+1}(h)-1} \int_{\tau_i}^{\tau_{i+1}} \{\delta^2 + h^2 + \gamma^h(\nu) + \gamma_i^h(\nu) + |\xi^h(\tau_i) - w(\tau_i)|^2\}\, d\nu,$$

where

$$\gamma^h(\nu) = \sum_{k=1}^{m} |u(\nu - \tau_k^u) - v^h(\nu - \tau_k^u)|^2, \quad \gamma_i^h(\nu) = \sum_{k=0}^{n} |x(\nu - \tau_k^x) - \xi^h(\tau_i - \tau_k^x)|^2.$$

Note that

$$\int_{t_j}^{t_{j+1}} \gamma^h(\nu)\, d\nu \le d_j^{(2)} \int_{t_{j-l-1}}^{t_j} |u(\nu) - v^h(\nu)|^2\, d\nu = d_j^{(2)} \sum_{k=j-l}^{j} \nu^{(k)}, \quad (23)$$

$$\int_{t_j}^{t_{j+1}} \gamma_i^h(\nu)\, d\nu \le d_j^{(3)}(h^2 + \delta^2). \quad (24)$$

In addition,

$$\nu^{(k)} = 0 \qquad k \in [-l : 0]. \quad (25)$$

Therefore, combining inequalities (22)–(24), we obtain the estimates

$$\varrho_j \le d_j^{(5)}\{h^2 + \delta^2 + \sum_{k=j-l}^{j} \nu^{(k)} + b_j + \alpha_j\}, \quad j \in [0 : j_*]. \quad (26)$$

One can easily see that the following estimates also hold:

$$|f_2(\cdot) - F_2(\cdot)|^2_{L_2(\Delta^{(j)};R^{n_2\times n_1})} \le d_j^{(5)}(h^2 + \delta^2), \quad j \in [0 : j_*]. \quad (27)$$

Here $d_j^{(1)}$–$d_j^{(5)}$ are some constants, which can be explicitly written. By lemma 3, (22), and (25), for $\delta \le h$, we have the inequalities

$$\varepsilon_0(t) \le b_0 \le c_0^* h, \quad t \in \Delta^{(0)}, \quad (28)$$

$$|x(t_1) - w(t_1)|^2 \leq \rho_A \alpha_0 + c_0^* h \leq c_* h^{2/3}. \tag{29}$$

Taking into account (25)–(28), for $h \in (0,1)$, we obtain

$$\varrho_0 \leq d_0^{(1)}\{h^2+\delta^2+b_0+h^{2/3}\} \leq d_0^* h^{2/3}, \qquad |f_2(\cdot)-F_2(\cdot)|^2_{L_2(\varDelta^{(0)};R^{n_2\times n_1})} \leq c_j^{(*)} h^2.$$

By virtue of condition 1, one can use lemma 1. Set $p = x$, $q = w$, $u_1 = u$, $u_2 = v^h$, $f_1(t) = f_1(t, u_t(s), x_t(s))$, $f_2(t) = f_2(t, x_t(s))$, $F_1(t) = f_1(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s)) + 2(\xi^h(\tau_i) - w(\tau_i))$, $F_2(t) = f_2(\tau_i, \xi^h_{\tau_i}(s))$ $t \in [\tau_i, \tau_{i+1})$. Then, assuming $a_1^{(0)} = d_0^* h^{2/3}$, $a_2^{(0)} = c_j^{(*)} h^2$, $a_3^{(0)} = c_0^* h$, $a_4^{(0)} = c_* h^{2/3}$, $\tilde{\alpha}_0 = \alpha_0 = c h^{2/3}$, we have

$$\nu^{(1)} = |u(\cdot) - v^h(\cdot)|^2_{L_2(\varDelta^{(0)};R^{n_1})} \leq \tilde{c}_1 h^{1/3} = c_1 g_1(h). \tag{30}$$

It means that inequality (20) holds for $j = 1$. Further, by using (29) and (30), we deduce that

$$b_1 = |x(t_1) - w(t_1)|^2 + c_1^{(1)}(h+\delta) + c_1^{(2)} \sum_{k=1-l}^{1} \nu^{(k)} \leq \tilde{c}_1^{(0)} h^{1/3} = c_1^{(0)} g_1(h).$$

Inequality (21) for $j = 1$ is also verified. It follows from (22) that

$$|x(t_j) - w(t_j)|^2 \leq b_{j-1} + \rho_A \alpha_{j-1}, \quad j \in [1 : j_* - 1]. \tag{31}$$

Consequently, in view of relations (31), as well as of the rule for definition b_j, we have the inequality

$$b_j \leq b_{j-1} + d_j\Big(h + \alpha_{j-1} + \sum_{k=j-l}^{j} \nu^{(k)}\Big), \quad d_j = \text{const} \in (0, +\infty). \tag{32}$$

Setting $a_1^{(j)} = d_j^{(4)}\{h^2 + \delta^2 + \sum_{k=j-l}^{j} \nu^{(k)} + a_3^{(j)} + \alpha_j\}$, $a_3^{(j)} = b_j$, $a_2^{(j)} = d_j^{(5)}(h^2 + \delta^2)$, $a_4^{(j)} = b_{j-1} + \rho_A \alpha_{j-1}$, $j \in [1 : j_*]$ for $j \geq 1$ in lemma 1 and taking into account inequalities (32), we obtain

$$\nu^{(j+1)} \leq c^{(j)}\{h^{1/2} + \Big(\sum_{k=j-l}^{j} \nu^{(k)}\Big)^{1/2} + b_{j-1}^{1/2} + \alpha_{j-1}^{1/2} + \alpha_j^{1/2}\} + b_j \alpha_j^{-1}, \ j \in [1 : j_*].$$

Here, we used lemma 3 and inequalities (27), (28), and (31)) for choosing values $a_i^{(j)}$. Now, to proof inequalities (20) and (21), one can use the proof by induction. The lemma is proved.

3 Example

The algorithm was tested by a model example. The following system

$$\begin{aligned} \dot{x}_1(t) &= 2x_1(t-1) + u(t) \\ \dot{x}_2(t) &= x_2(t-1) + x_1(t) + u(t-1), \quad t \in T = [0, 2], \end{aligned} \tag{33}$$

with initial conditions $x_0(s) = y_0(s) = 1$, $u(s) = 0$ for $s \in [-1, 0]$ and control $u(t) = t$ was considered. The solution $x(t) = \{x_1(t), x_2(t)\}$ of system (33) was calculated analytically. During the experiment, we assumed that $\xi^h(\tau_i) = x_1(\tau_i) + h$. As a model, we took the system (9), which has the form

$$\begin{aligned} \dot{w}^{(0)}(t) &= 2\xi_1^h(\tau_i - 1) + v_i^h + 2(\xi_1^h(\tau_i) - w^{(0)}(\tau_i)) \quad \text{for} \quad t \in [\tau_i, \tau_{i+1}) \\ \dot{w}^{(1)}(t) &= \xi_2^h(\tau_i - 1) + \xi_1^h(\tau_i) + v^h(\tau_i - 1) + 2(\xi_2^h(\tau_i) - w^{(1)}(\tau_i)), \end{aligned} \tag{34}$$

with the initial condition $w^{(0)}(s) = w^{(1)}(s) = 1$, for $s \in [-1, 0]$. Here $v^h(\tau_i) = v_i^h$ for $t \in [\tau_i, \tau_{i+1})$, $i \geq 0$, $v^h(s) = 0$ for $s \in [-1, 0)$. The controls v_i^h in model (34) were calculated by the following formula (see (11))

$$v_i^h = \arg\min\{2l_i v + \alpha_j |v|^2 : |v| \leq K\},$$

where $l_i = w^{(0)}(\tau_i) - \xi_1^h(\tau_i)$.

In figures 1 and 2 the results of calculations are presented for the case when $\delta = 10^{-4}$, $\alpha_0 = Ch^{2/3}$, $\alpha_1 = Ch^{2/9}$, $C = 0.2$, $K = 10$. Fig. 1 corresponds to the case when $h = 0.001$, fig. 2 — $h = 0.02$. In these figures the solid (dashed) lines represent the model control $v^h(\cdot)$ (the real control $u(\cdot)$). The equations were solved by the Euler method with step δ.

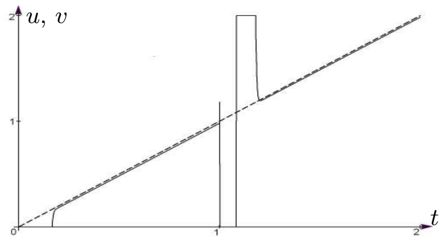

Fig. 1.

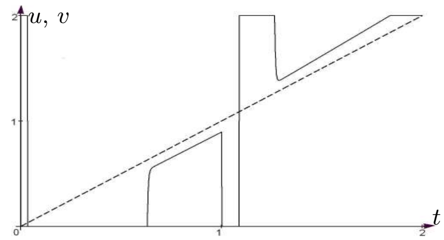

Fig. 2.

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