ALGEBRAIC PROPERTY OF ROUGH IMPLICATION BASED ON INTERVAL STRUCTURE

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Abstract: Due to the shortage of rough implication in [4] ~ [6], rough set and rough implication operators are redefined by using interval structure in [7], the shortages have been e improved. We have investigated the characteristics of the rough implication, and also point out that the good logic property of the rough implication in [7]. In this paper, we will study the algebraic properties of the rough implication in depth.

Key words: Rough Logic, Algebraic Property, Rough Implication, Approximation Spaces

1. INTRODUCTION

Rough set theory, introduced by Zdzislaw Pawlak in the early 1980s [1-3], is a new mathematical tool to deal with many problems such as vagueness, uncertainty, incomplete data and reasoning. Now there are lots of papers about rough logic idea and its abroad application [1~9], but some rough implication operators exist defects, for instance, $B^c \to A^c = A \to B$ doesn't hold in [4], $A \to A$ is not Theorem in [5,6], etc.. In order to eliminate those defects we redefine rough set system, and new rough operators such as intersection, union, complement and implication are expressed by using interval structure in [7]. The characteristics of this implication were investigated, and logic properties of rough implication were pointed out in [7]. Further, we will study the algebraic properties of the rough implication in this paper.

2. ROUGH SET THEORY

Definition2.1 Let U be the universe set and R be an equivalent relation on U. A pair

(U, R) is called an approximate space. If $X \subseteq U$ is an arbitrary set, then two approximations are formally defined as follows:

$\underline{X} = \{x \mid x \in U, [x]_R \subseteq X\}$, $\quad \overline{X} = \{x \mid x \in U, [x]_R \cap X \neq \phi\}$.

Where $[x]_R$ is an equivalent class containing x. $\underline{X}$ is called lower approximation of X, $\overline{X}$ is called upper approximation of X. The approximate set X lies between its lower and upper approximations: $\underline{X} \subseteq X \subseteq \overline{X}$.

We get $-\overline{X} \subseteq -X \subseteq -\underline{X}$, where, $Z \subseteq U$ and $-Z$ is the complement of Z in U.

For each $X \subseteq U$, a rough set is a pair $\langle \underline{X}, \overline{X} \rangle$. We denote the empty set ϕ by $\langle \underline{\phi}, \overline{\phi} \rangle = \langle \phi, \phi \rangle$, the universe set U by $\langle \underline{U}, \overline{U} \rangle = \langle U, U \rangle$ and the power set of U by $\Re(U)$.

Definition 2.2 Let $A, B \in \Re(U)$, the inclusion relation of two rough sets is defined by $A \tilde{\subset} B$ if and only if $\overline{A} \subseteq \overline{B}$ and $\underline{A} \subseteq \underline{B}$;

The equivalent relation of two rough sets is defined by

$A = B$ if and only if $\overline{A} = \overline{B}$ and $\underline{A} = \underline{B}$.

Definition 2.3 The intersection of two rough sets A and B is a rough set in approximate space, and is defined by $A \cap B = \langle \underline{A} \cap \underline{B}, \overline{A} \cap \overline{B} \rangle$,

The union of two rough sets is a rough set in approximate space, and is defined by $A \cup B = \langle \underline{A} \cup \underline{B}, \overline{A} \cup \overline{B} \rangle$,

The complement of A is a rough set in approximate space, and is defined by $A^c = \langle -\overline{A}, -\underline{A} \rangle$,

The pseudo complement of A is a rough set in approximate space, and is defined by $A^* = \langle -\underline{A}, -\underline{A} \rangle$,

Where $X \subseteq U$, $-X$ is the complement of X in U.

Theorem 2.4 Suppose $A, B \in \Re(U)$, then

$\underline{A \cap B} = \underline{A} \cap \underline{B}$, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$;

$\underline{A \cup B} \supseteq \underline{A} \cup \underline{B}$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. Theorem 2.4 follows from [1] ~ [3] and [8] ~ [9].

Theorem 2.5 If A^c is the complement of A in U, A^* is the pseudo complement of A in U, then

(1) $A^c \subseteq A^*$; (2) $A^{**} \subseteq A^{c*}$; (3) $A^c \cup A^* = A^*$, $A^c \cap A^* = A^c$;

(4) $A^{c*c} = A^{c**} = \langle -\overline{A}, -\overline{A} \rangle$; (5) $A^{cc*} = A^{*c*} = A^{***} = A^{**c} = A^{*cc} = A^*$;

(6) $A^{ccc} = A^c$; (7) $A^{c*c*} = A^{c*}$.

Proof. Theorem 2.5 can be proved easily from Definition 2.3.

Theorem 2.6 Let $A, B \in \Re(U)$, then,

$(A \cap B)^c = A^c \cup B^c$; $\quad (A \cup B)^c = A^c \cap B^c$;

$(A \cap B)^* = A^* \cup B^*$; $\quad (A \cup B)^* = A^* \cap B^*$.

Proof. Theorem 2.6 is easy to be proved by Definition 2.3.

3. ALGEBRAIC PROPERTIES OF ROUGH IMPLICATION

We redefine the implication operator in [7], which to improve the shortage of

rough implication in [4]~[6]. In this section, we will directly cite the definition implication operator $\rightarrow$, and will investigate its algebraic properties.

Definition 3.1 Let $mng\ (\varphi) = \langle \underline{A}, \overline{A} \rangle$, $mng(\psi) = \langle \underline{B}, \overline{B} \rangle$, $mng(\beta) = \langle \underline{C}, \overline{C} \rangle$, and *mng* is a bijection, for any $\varphi, \psi, \beta, 0, 1 \in P$, we have

$mng(\varphi \wedge \psi) = \langle \underline{A} \cap \underline{B}, \overline{A} \cap \overline{B} \rangle$; $mng(\varphi \vee \psi) = \langle \underline{A} \cup \underline{B}, \overline{A} \cup \overline{B} \rangle$;

$mng(\varphi^c) = \langle -\overline{A}, -\underline{A} \rangle$; $mng(\varphi^*) = \langle -\underline{A}, -\underline{A} \rangle$; $mng(0) = \langle \phi, \phi \rangle$;

$mng(\varphi^{c*}) = \langle \overline{A}, \overline{A} \rangle$; $mng(\varphi^{c*c}) = \langle -\overline{A}, -\overline{A} \rangle$; $mng(1) = \langle U, U \rangle$.

$(\varphi \vee \psi)^c = \varphi^c \wedge \psi^c$; $(\varphi \wedge \psi)^c = \varphi^c \vee \psi^c$; $(\varphi \vee \psi)^* = \varphi^* \wedge \psi^*$;

$(\varphi \wedge \psi)^* = \varphi^* \vee \psi^*$; $\varphi^{c*c} = \varphi^{c**}$; $\varphi^{ccc} = \varphi^c$;

$\varphi^{c*c*} = \varphi^{c*}$; $\varphi^{cc*} = \varphi^{*c*} = \varphi^{***} = \varphi^{**c} = \varphi^{*cc} = \varphi^*$.

$mng(\varphi \rightarrow \psi) = mng(\varphi^c \vee \psi \vee (\varphi^* \wedge \psi^{c*})) = \langle -\overline{A} \cup \underline{B} \cup (\overline{B} \cap -\underline{A}), -\underline{A} \cup \overline{B} \rangle$
(I)

Theorem 3.2 $(P, \vee, \wedge, ^c, 0, 1)$ is a boundary lattice.

Proof. Theorem3.2 is easy to prove from definition 3.1 and [7].

Theorem 3.3 Let $A, B \in \Re(U)$, the following are equivalent:

(1) $A \rightarrow B = \langle -\overline{A} \cup \underline{B} \cup (\overline{B} \cap -\underline{A}), -\underline{A} \cup \overline{B} \rangle$;

(2) $A \rightarrow B = \langle (-\underline{A} \cup \underline{B}) \cap (-\overline{A} \cup \overline{B}), -\underline{A} \cup \overline{B} \rangle$.

Proof. Theorem 3.3 is easy to be proved from (I).

Proposition 3.4 Suppose $(P, \vee, \wedge, ^c, 0, 1)$ is called a boundary lattice which is inverse ordered involution, and $\rightarrow$ is rough implication operator, the following are satisfied:

(IA.1) $\varphi \rightarrow (\psi \rightarrow \beta) = \psi \rightarrow (\varphi \rightarrow \beta)$ (IA.2) $\varphi \rightarrow \varphi = 1$

(IA.3) $\varphi \rightarrow \psi = \psi^c \rightarrow \varphi^c$ (IA.4) if $\varphi \rightarrow \psi = \psi \rightarrow \varphi = 1$, then $\varphi = \psi$

(IA.5)
$\varphi \vee \psi \rightarrow \beta = (\varphi \rightarrow \beta) \wedge (\psi \rightarrow \beta)$ (IA.6) $\varphi \wedge \psi \rightarrow \beta = (\varphi \rightarrow \beta) \vee (\psi \rightarrow \beta)$

Proof. The formulas can be proved by Theorem 3.3 and (I).

Proof of (IA.1)

$\varphi \rightarrow (\psi \rightarrow \beta) = \varphi^c \vee (\psi \rightarrow \beta) \vee (\varphi^* \wedge (\psi \rightarrow \beta)^{c*})$

$= \varphi^c \vee (\psi^c \vee \beta \vee (\psi^* \wedge \beta^{c*})) \vee (\varphi^* \wedge (\psi^c \vee \beta \vee (\psi^* \wedge \beta^{c*}))^{c*})$

$= \varphi^c \vee \psi^c \vee \beta \vee (\psi^* \wedge \beta^{c*}) \vee (\varphi^* \wedge (\psi^* \vee \beta^{c*} \vee (\psi^* \wedge \beta^{c*})))$

$= \varphi^c \vee \psi^c \vee \beta \vee (\psi^* \wedge \beta^{c*}) \vee (\varphi^* \wedge \psi^*) \vee (\varphi^* \wedge \beta^{c*}) \vee (\varphi^* \wedge \psi^* \wedge \beta^{c*})$

$\psi \rightarrow (\varphi \rightarrow \beta) = \psi^c \vee (\varphi \rightarrow \beta) \vee (\psi^* \wedge (\varphi \rightarrow \beta)^{c*})$

$= \psi^c \vee (\varphi^c \vee \beta \vee (\varphi^* \wedge \beta^{c*})) \vee (\psi^* \wedge (\varphi^c \vee \beta \vee (\varphi^* \wedge \beta^{c*}))^{c*})$

$= \psi^c \vee \varphi^c \vee \beta \vee (\varphi^* \wedge \beta^{c*}) \vee (\psi^* \wedge (\varphi^* \vee \beta^{c*} \vee (\varphi^* \wedge \beta^{c*})))$

$= \psi^c \vee \varphi^c \vee \beta \vee (\varphi^* \wedge \beta^{c*}) \vee (\psi^* \wedge \varphi^*) \vee (\psi^* \wedge \beta^{c*}) \vee (\psi^* \wedge \varphi^* \wedge \beta^{c*})$

$= \varphi^c \vee \psi^c \vee \beta \vee (\psi^* \wedge \beta^{c*}) \vee (\varphi^* \wedge \psi^*) \vee (\varphi^* \wedge \beta^{c*}) \vee (\varphi^* \wedge \psi^* \wedge \beta^{c*})$

Hence, $\varphi \to (\psi \to \beta) = \psi \to (\varphi \to \beta)$

Proof of (IA.2) Obviously, $\varphi \to \varphi = 1$

Proof of (IA.3)

$\varphi^c \to \psi^c = \varphi^{cc} \vee \psi^c \vee (\varphi^{c*} \wedge \psi^{cc*}) = \psi^c \vee \varphi \vee (\psi^* \wedge \varphi^{c*})$

$\psi \to \varphi = \psi^c \vee \varphi \vee (\psi^* \wedge \varphi^{c*})$

Hence, $\varphi \to \psi = \psi^c \to \varphi^c$.

Proof of (IA.4)

Because of $A \to B = \langle (-\underline{A} \cup \underline{B}) \cap (-\overline{A} \cup \overline{B}), -\underline{A} \cup \overline{B} \rangle$ $A \to B = U$ iff $\mathrm{A} \to \mathrm{B} = \langle (-\underline{\mathrm{A}} \cup \underline{\mathrm{B}}) \cap (-\overline{\mathrm{A}} \cup \overline{\mathrm{B}}), -\underline{\mathrm{A}} \cup \overline{\mathrm{B}} \rangle = \langle \mathrm{U}, \mathrm{U} \rangle$ iff $-\underline{\mathrm{A}} \cup \underline{\mathrm{B}} = \mathrm{U}, -\overline{\mathrm{A}} \cup \overline{\mathrm{B}} = \mathrm{U}, -\underline{\mathrm{A}} \cup \overline{\mathrm{B}} = \mathrm{U}$ iff $\underline{A} \subseteq \underline{B}$ and $\overline{A} \subseteq \overline{B}$ iff $A \tilde{\subseteq} B$.

Analogously we have $B \to A = U$ iff $B \tilde{\subseteq} A$.

Hence, (IA.4) is proved.

Proof of (IA.5)

$(\varphi \to \beta) \wedge (\psi \to \beta) = (\varphi^c \vee \beta \vee (\varphi^* \wedge \beta^{c*})) \wedge (\psi^c \vee \beta \vee (\psi^* \wedge \beta^{c*}))$

$= (\varphi^c \wedge \psi^c) \vee (\varphi^c \wedge \beta) \vee (\varphi^c \wedge \psi^* \wedge \beta^{c*}) \vee (\beta \wedge \psi^c) \vee \beta \vee (\beta \wedge \psi^* \wedge \beta^{c*})$

$\vee (\varphi^* \wedge \psi^c \wedge \beta^{c*}) \vee (\beta \wedge \varphi^* \wedge \beta^{c*}) \vee (\varphi^* \wedge \psi^* \wedge \beta^{c*})$

$= (\varphi^c \wedge \psi^c) \vee \beta \vee (\varphi^* \wedge \psi^* \wedge \beta^{c*})$

$(\varphi \vee \psi) \to \beta = (\varphi \vee \psi)^c \vee \beta \vee ((\varphi \vee \psi)^* \wedge \beta^{c*}) = (\varphi^c \wedge \psi^c) \vee \beta \vee (\varphi^* \wedge \psi^* \wedge \beta^{c*})$

Hence, $\varphi \vee \psi \to \beta = (\varphi \to \beta) \wedge (\psi \to \beta)$.

Proof of (IA.6)

$(\varphi \wedge \psi) \to \beta = (\varphi \wedge \psi)^c \vee \beta \vee ((\varphi \wedge \psi)^* \wedge \beta^{c*}) = (\varphi^c \vee \psi^c) \vee \beta \vee ((\varphi^* \vee \psi^*) \wedge \beta^{c*})$

$= (\varphi^c \vee \psi^c) \vee \beta \vee (\varphi^* \wedge \beta^{c*}) \vee (\psi^* \wedge \beta^{c*})$

$(\varphi \to \beta) \vee (\psi \to \beta) = (\varphi^c \vee \beta \vee (\varphi^* \wedge \beta^{c*})) \vee (\psi^c \vee \beta \vee (\psi^* \wedge \beta^{c*}))$

$= (\varphi^c \vee \psi^c) \vee \beta \vee (\varphi^* \wedge \beta^{c*}) \vee (\psi^* \wedge \beta^{c*})$

Hence, $\varphi \wedge \psi \to \beta = (\varphi \to \beta) \vee (\psi \to \beta)$.

The poor is complete.

Proposition 3.5 Suppose $(P, \vee, \wedge, {}^c, 0, 1)$ is called a boundary lattice which is inverse ordered involution, and $\to$ is rough implication operator which is expressed by interval structure, then $(\varphi \to \psi) \to \psi \neq (\psi \to \varphi) \to \varphi$

Proof.

$(\varphi \to \psi) \to \psi = (\varphi^c \vee \psi \vee (\varphi^* \wedge \psi^{c*}))^c \vee \psi \vee ((\varphi^c \vee \psi \vee (\varphi^* \wedge \psi^{c*}))^* \wedge \psi^{c*})$

$= (\varphi \wedge \psi^c \wedge (\varphi^{*c} \vee \psi^{c*c})) \vee \psi \vee ((\varphi^{c*} \wedge \psi^* \wedge (\varphi^{**} \vee \psi^{c**})) \wedge \psi^{c*})$

$= (\varphi \wedge \psi^c \wedge \varphi^{*c}) \vee (\varphi \wedge \psi^c \wedge \psi^{c*c}) \vee \psi \vee (\varphi^{c*} \wedge \psi^* \wedge \varphi^{**} \wedge \psi^{c*}) \vee (\varphi^{c*} \wedge \psi^* \wedge \psi^{c**} \wedge \psi^{c*})$

$= (\psi^c \wedge \varphi^{*c}) \vee (\varphi \wedge \psi^{c*c}) \vee \psi \vee (\psi^* \wedge \varphi^{**} \wedge \psi^{c*})$

Analogously, $(\psi \to \varphi) \to \varphi = (\varphi^c \wedge \psi^{*c}) \vee (\psi \wedge \varphi^{c*c}) \vee \varphi \vee (\varphi^* \wedge \psi^{**} \wedge \varphi^{c*})$

Hence, $(\varphi \to \psi) \to \psi \neq (\psi \to \varphi) \to \varphi$ (except for $\varphi = \psi$).

Remark If $(P, \vee, \wedge, {}^c, 0, 1)$ is lattice implicative algebra [11~13], then it is satisfied Proposition 3.4 (IA.1) ~ (IA.6) and $(x \to y) \to y = (y \to x) \to x$. Since the equivalence of lattice implicative algebra is normal *FI*-algebra [11,

[14], the operator $\rightarrow$ is not satisfied $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$. Hence, $(P, \vee, \wedge, ^c, 0, 1)$ is not lattice implicative algebra, but it is *FI*-algebra [11, 14].

4. CONCLUSION

The study of rough implication operators is the emphasis and difficulty in the field of rough logic. Due to definition the shortages of the rough implication operator in [4] ~ [6], we can not imply $B^c \rightarrow A^c = A \rightarrow B$ in [4], i.e. the inversely negative proposition and original proposition are not equivalent, and A→A isn't Theorem in [5, 6], etc... We redefine the rough intersection, rough union, rough complement and rough implication operator from the view of interval structure, which their relations and properties have been investigated in [7]. In this paper, we_study the algebraic properties of the rough implication in a deep way, and also point out that $(P, \vee, \wedge, ^c, 0, 1)$ is not lattice implicative algebra, but it is *FI*-algebra, because the formula $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ doesn't hold.

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